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Comparaison entre la transmission de l'Information en Optique et en Radioélectricité *

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SOMMAIRE. — *On compare les propriétés des transformations qui traduisent le comportement des diverses parties d'un dispositif optique avec celles d'un dispositif radioélectrique courant.*

La notion de bande passante appliquée à l'optique est discutée en liaison avec celle de pouvoir séparateur ou de limite de résolution. Des expériences de principe permettent d'espérer des améliorations du pouvoir séparateur en transposant à l'optique des idées d'origine radioélectrique sont décrites.

Enfin on discute l'introduction d'un élément d'imprécision jouant en optique le rôle du bruit de fond. On est alors conduit à la transposition à l'optique des principaux résultats de la théorie de l'information.

ZUSAMMENFASSUNG. — *Man kann die Übertragungseigenschaften der verschiedenen Teile einer optischen Einrichtung mit denen der üblichen Nachrichtentechnik vergleichen.*

Der Begriff der Bandbreite lässt sich auch auf die Optik anwenden und mit dem der Trennschärfe und Auflösungsgrenze in Verbindung bringen. Versuche in dieser Richtung lassen erwarten, dass man zu einer Verbesserung der Trennschärfe kommt, wenn man die ursprünglich aus der Nachrichtentechnik stammenden Überlegungen auf die Optik überträgt. In diesem Sinne kann man eine "Ungenauigkeitsgrösse" einführen, die die Rolle des Störpegels übernimmt. Dadurch kommt man zu der Übertragung der grundsätzlichen Ergebnisse der Informationstheorie auf die Optik.

SUMMARY. — *The properties of the transformations which express the behaviour of the different parts of an optical system are compared with those of a radio-frequency circuit.*

The idea of " band-pass " applied to optics is discussed, together with that of limit of resolution. Ideal experiments are described offering the possibility of improvements in resolving power by applying to optics ideas originating in radio.

Finally the introduction of an element of uncertainty corresponding in optics to that of noise is considered. This leads to the application of information theory to optics.

Nous nous proposons de comparer la transmission de l'Information en radioélectricité et la transmission de l'Information en optique. Cette comparaison suggère certaines solutions qui peuvent rendre des services en optique.

I. Comparaison générale entre la transmission de l'information en radioélectricité et en optique. — Raisonnons sur le modèle d'une transmission par modulation d'amplitude (voir fig. 1). En Radio et en Optique, on peut, en schématisant, distinguer :

a) *une source* caractérisée par une fonction $S(t)$ (du temps en radio) ou $S(\mathrm{M})$ (du point M, en optique). $S(\mathrm{M})$ est la densité d'amplitude sur la source (cette densité existe sauf dans le cas d'une incohérence stricte).

b) *un dispositif de liaison* } filtre linéaire F_1 : réponse percussionnelle $r(t)$ [ou $r(\overrightarrow{\mathrm{M}})$] et gain $g(\nu)$ [ou $g(\overrightarrow{\Omega})$] }. En optique, ce dispositif est le condenseur ;

c) *un modulateur*, dont nous traduirons approximativement l'effet en disant qu'il multiplie « la porteuse » par le « message » $T(t)$ (ou $T(\mathrm{M})$) ; en optique, T est la transparence de l'objet à examiner.

d) *un second filtre linéaire* F_2 (R et G).

e) *un dispositif de détection* supposé quadratique.

Le schéma I *bis* (fig. 1) concerne un exemple très simple dans lequel on observe la variation d'une résistance $r(t)$ en suivant celle de la différence de potentiel à ses bornes lorsqu'elle est parcourue par un courant I.

On peut faire les remarques suivantes :

1°. L'existence de phénomènes de fluctuation donne à S un certain caractère aléatoire. Ce caractère aléatoire est très peu marqué en radio où $S(t)$ est une fonction à corrélation très large tandis qu'en optique, $S(\mathrm{M})$ possède une corrélation microscopique. La source $S(\mathrm{M})$ est incohérente, tandis que $S(t)$ se rapproche du cas de l'éclairage cohérent. Pour avoir, en radio, l'équivalent d'une source d'optique, il faudrait prendre un générateur de bruit à très large bande.

2°. Le problème de radio est marqué par les propriétés particulières du paramètre temps. Les réponses percussionnelles des filtres y sont nulles pour $t < 0$;

(*) Nous résumons dans e présent article deux communications présentées au Colloque de Florence « Problems in contemporary Optics », septembre 1954, dont les titres sont :

a) application de l'analyse harmonique et de la théorie de l'Information à l'étude de la correspondance objet-image en optique. Comparaison avec la correspondance signal-réponse en radioélectricité par A. BLANC-LAPIERRE ;

b) sur deux expériences de principe illustrant quelques possibilités d'amélioration de la limite de résolution d'un instrument d'optique par A. BLANC-LAPIERRE, M. PERROT et G. PÉRI.

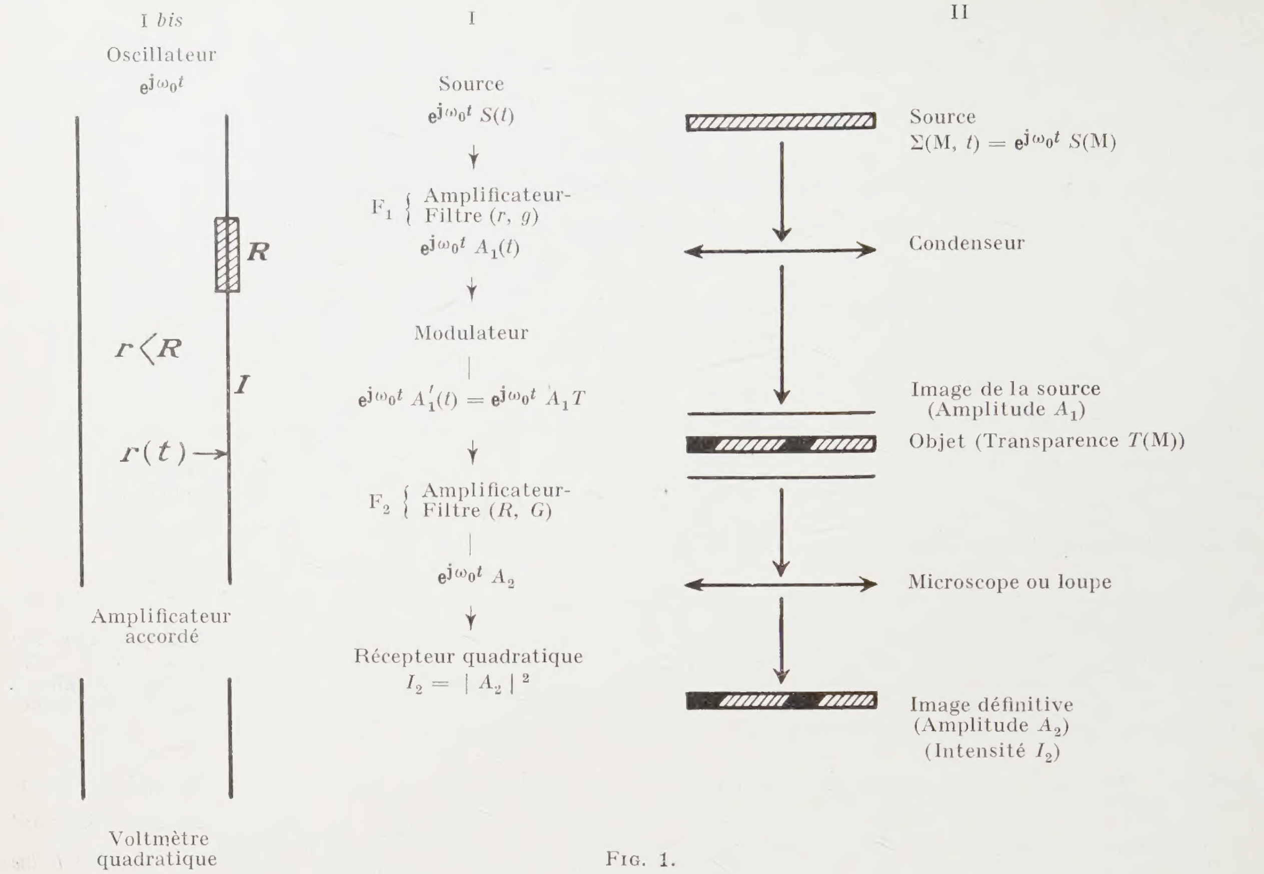

FIG. 1.

il n'y a rien d'équivalent en optique. A cause de l'homogénéité du temps, les amplificateurs en radio sont *exactement* représentés par des filtres linéaires ; cela n'est souvent qu'une *bonne approximation* en optique, la distribution d'amplitude complexe dans une tache de diffraction pouvant ne pas être parfaitement invariante (en module et en argument) si on déplace le point-objet dans le champ.

3°. On peut noter que, d'un certain point de vue, notre schéma d'optique présente un caractère un peu artificiel car il correspond à un monochromatisme strict ; or, une étude complète de la cohérence ou des propriétés statistiques de la lumière ne peut se faire dans le cadre d'un monochromatisme parfait [1].

II. L'approximation des filtres linéaires en optique

[2] [8]. — Soit un système optique où nous ne prenons en considération que les aberrations dues à la diffraction (on opère en lumière monochromatique). Si A(M) est l'amplitude au point M du plan-objet, l'amplitude A'(M$'$) au point M$'$ du plan-image, se déduit en général de A(M) avec une bonne approximation par un filtre linéaire

$$(1) \qquad A'(M') = \int A(M)\, R(M' - M)\, dM$$

ou

$$a'[\Omega] = G(\Omega)\, a(\Omega)$$

R : réponse percussionnelle ; G, gain ; a et a', transformées de FOURIER de A et A' ; Ω fréquence spatiale. Si $\mathfrak{T}(m)$ est la transparence complexe dans le plan de la pupille de diffraction, pour un choix convenable des paramètres, on peut poser

$$(2) \qquad G(\Omega) = \mathfrak{T}(m) \quad \text{pour} \quad m = \Omega$$

G est nul si Ω est extérieur à un domaine (α) lié à la forme de la pupille.

III. Solution générale du problème de la correspondance $T \to I_2 = |A_2|^2$.

— Prenons le problème d'optique (schéma II de la figure 1). Supposons la source incohérente et macroscopiquement uniforme. Soient r et g la réponse percussionnelle et le gain du condenseur, R et G les mêmes fonctions pour le microscope (²). La cohérence relative à l'éclairage de l'objet est définie par la covariance [1]

$$(3) \qquad \Gamma(M_1, M_2) = \int r(M_1 - M)\, r^*(M_2 - M)\, dM = \int |g(\Omega)|^2 e^{2\pi i \vec{\Omega}(\vec{M_1} - \vec{M_2})}\, d\Omega.$$

(²) Comme déjà indiqué, S a un caractère aléatoire ; mais nous ne tenons pas compte, pour l'instant, de tout élément d'imprécision introduit entre la source et le plan-image.

La distribution d'intensité dans l'image est

$$(4) \quad I_2(\mathrm{M}) = \iint R(\mathrm{M}-\mathrm{M}_1)\, R^*(\mathrm{M}-\mathrm{M}_2)\, T(\mathrm{M}_1)\, T^*(\mathrm{M}_2) \times \Gamma(\mathrm{M}_1-\mathrm{M}_2)\, d\mathrm{M}_1\, d\mathrm{M}_2.$$

La transformée de FOURIER $i_2[\Omega]$ de $I_2[\mathrm{M}]$ est

$$(5) \quad i_2[\Omega] = \int E(\Omega, \Omega_2)\, t(\Omega_2)\, t^*(\Omega_2-\Omega)\, d\Omega_2$$

où t (transformée de FOURIER de T) caractérise l'objet et où E caractérise l'appareillage (condenseur + microscope) ; E est défini par

$$(6) \quad E(\Omega, \Omega_2) = \int G(\Omega_1)\, G^*(\Omega_1-\Omega_2)\, d(\Omega_1-\Omega_2)\, d\Omega_1$$

$d = |g|^2$ est la transformée de FOURIER de $\Gamma(\mathrm{M}_1-\mathrm{M}_2)$; les formules (4), (5), (6) sont, à des changements de notations près, identiques à celles qui ont été antérieurement données par H. H. HOPKINS [3].

Dans le cas de la cohérence, on passe de T à I_2 par l'ensemble des deux opérations suivantes effectuées successivement dans l'ordre indiqué ci-dessous.

1°. un filtrage linéaire [gain $G(\Omega)$] qui fait passer de T à A_2,

2°. une élévation au carré (détection) qui fait passer de A_2 à I_2.

Dans le cas de l'incohérence il faut permuter l'ordre de ces deux opérations et effectuer successivement :

1°. une élévation au carré $T \to |T|^2$,

2°. un filtrage linéaire de gain :

$\gamma(\Omega) = \int G(\Omega')\, G^*(\Omega'-\Omega)\, d\Omega$ qui fait passer de $|T|^2$ à I_2.

$\gamma(\Omega)$ n'est différent de zéro que dans un domaine $\mathcal{L}$ borné entourant l'origine qui joue, dans l'éclairage incohérent, le rôle d'une *bande passante* [en éclairage cohérent ce rôle était tenu par (α)].

IV. **Propriétés de la voie de communication objet-image en éclairage incohérent et en l'absence de tout bruit** [4].

a) Toute composante de $|T|^2$ de fréquence Ω extérieure à $\mathcal{L}$ (onde Θ_1) est éliminée ; c'est cela qui crée la limite de résolution.

b) Toute composante de $|T|^2$ de fréquence Ω intérieure à $\mathcal{L}$ (onde Θ_2) est transmise ; mais $\gamma(\Omega)$ est fonction de Ω ; il y a distorsion d'amplitude.

En transposant à l'optique des idées courantes en radioélectricité on peut chercher à améliorer la situation dans l'une ou l'autre des directions suivantes :

1) *On cherche à rétablir dans $\mathcal{L}$ une transmission correcte, c'est-à-dire, une transmission à gain constant.* Pour cela, il suffit, en principe, de remplacer I_2 par sa transformée dans un filtre de gain $1/\gamma(\Omega)$ dans $\mathcal{L}$. BLANC-LAPIERRE, PERROT et DUMONTET ont proposé une méthode pratique pour réaliser approximativement cette correction dans des problèmes unidimensionnels [5].

A. MARÉCHAL et CROCE ont indiqué le principe d'une méthode permettant de réaliser cette correction en éclairant en lumière cohérente, sous certaines conditions, la photographie obtenue en éclairage incohérent (6).

2) *Tout objet qui ne contient que des composantes Θ_1 n'est pas transmis.* A. BLANC-LAPIERRE, M. PERROT et G. PERI [7] ont effectué des expériences de principe, à grande échelle, qui montrent que, dans certaines circonstances, l'existence de tels objets peut être décelée et certains renseignements sur leur structure obtenus. Pour avoir une image, il faut faire apparaître un système d'ondes Θ_2 lié à l'objet. On peut y parvenir par l'un ou l'autre des deux procédés suivants :

a) *un changement de fréquence* obtenu en superposant à l'objet un autre objet de structure périodique ;

b) *une détection.* Si T n'a pas de basses fréquences, T^2 en aura en général. On remplace la transparence T par son carré en faisant en sorte que l'objet soit traversé deux fois par la lumière dans des conditions convenables.

V. **La voie de communication optique en présence de bruit** (éclairage incohérent). — Au cours de ces dernières années divers articles ont été publiés sur l'application à l'optique de la théorie de l'information [8], [9], [10], [11], [12].

L'existence d'une bande passante limitée permet de transposer à l'optique les résultats de la théorie de l'information relatifs au nombre de degrés de liberté ; l'introduction du bruit conduit à la quantification de ces degrés de liberté et l'essentiel de la théorie de l'information peut être étendu. Une difficulté mathématique vient de ce qu'en éclairage incohérent l'objet et l'image sont caractérisés par des fonctions positives. Cette difficulté peut, en particulier, être simplement éludée si, posant $\theta = |T|^2$, on fait l'hypothèse $|\theta - \overline{\theta}| \ll \overline{\theta}$, c'est-à-dire si on se limite à des objets très faiblement contrastés. On peut alors admettre qu'en première approximation il n'y a plus de restrictions sur $\theta' = \theta - \overline{\theta}$ et raisonner sur θ' et sur $I_2' = I_2 - \overline{I_2}$. Il faut faire des hypothèses sur les propriétés que nous attribuerons au bruit B(M) (3). Ces hypothèses comportent une part d'arbitraire. Dans une approximation grossière, on peut supposer qu'un bruit gaussien, indépendant de l'objet, stationnaire en M, de spectre uniforme (densité constante b_1) est appliqué au niveau de l'objet, c'est-à-dire à l'entrée du microscope, et qu'un bruit semblable est appliqué à la sortie du microscope (densité spectrale constante b_2) (4). Il revien-

(3) (bruit = ensemble des causes d'imprécision : fluctuations dans l'éclairage de l'objet, diffusions accidentelles, fluctuations dues à l'appareil d'observation, fluctuations dues à la nature même de la lumière, etc.).

(4) G. TORALDO DI FRANCIA a indiqué à Florence le résultat d'une étude dans laquelle il suppose que le bruit optique est gaussien et a un écart-type lié à l'éclairement de l'objet. Cette hypothèse serre probablement la réalité de plus près que celles dont on donne ici les conséquences.

drait au même d'appliquer à l'entrée un bruit fictif, stationnaire en M, gaussien, de densité spectrale

$$(7) \qquad b(\Omega) = b_1 + b_2/|\ G\ |^2.$$

Considérons le cas où l'ensemble des objets possibles forme une famille aléatoire telle que, dans le champ de l'appareil, on puisse considérer θ' comme une fonction aléatoire stationnaire de M et supposons la « puissance moyenne » $\overline{\theta'^2}$ imposée ; la capacité C de la ligne optique de transmission (quantité maximum d'information transmissible par unité d'aire du plan-objet) s'obtient de la façon suivante [12]. Soit un système de coordonnées cartésiennes ξ, η, ζ [$(\xi, \eta) = \Omega$ et $O\zeta$ vertical et dirigé vers le haut] ; considérons la surface π d'équation $\zeta = b(\Omega)$ comme limitant une cuvette (voir fig. 2) ; versons-y un liquide de volume $\overline{\theta'^2}$. Soit $\zeta = D_0$ le niveau de la surface libre du liquide.

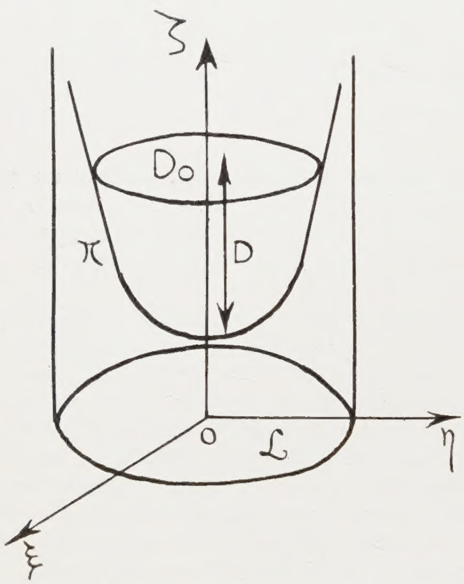

FIG. 2.

Posons $D = 0$ si $D_0 < b$ et $D = D_0 - b$ si $D_0 > b$. On a

$$(8, a) \qquad C = \frac{1}{2} \iint_{\mathcal{L}} \operatorname{Log}\left(1 + \frac{D}{b}\right) d\Omega,$$

soit, pour $b_2 = 0$,

$$(8, b) \qquad C = \frac{L}{2} \operatorname{Log}\left(1 + \frac{\overline{\theta'^2}}{\overline{B^2}}\right)$$

où L est l'aire de $\mathcal{L}$ dans $\xi \times \eta$ évaluée en unités convenables et $\overline{B^2}$ la puissance moyenne du bruit.

La forme simple (8, *b*) montre bien les facteurs qui interviennent dans la capacité :

a) L mesure l'extension de la bande passante,

b) $\overline{\theta'^2}$ mesure la dispersion du signal autour de sa valeur moyenne,

c) $\overline{B^2}$ mesure l'importance du bruit.

La notion de pouvoir séparateur ignore les facteurs *b*) et *c*) pourtant essentiels si on veut juger la qualité de la transmission.

Exemple. — Soit une pupille carrée de côté m. Supposons le plan-objet situé à une distance l. La capacité C est définie par le fait que la quantité d'information associée à une petite surface Σ de l'objet sera égale à $C\Sigma/l^2$. En supposant la formule (8, *b*) valable, on a

$$(9) \quad C = \frac{2\,m^2}{\lambda^2} \operatorname{Log}\left[1 + \overline{\theta'^2}/\overline{B^2}\right] = \frac{2\,m^2}{\lambda^2} \operatorname{Log} \frac{\overline{B^2} + \overline{\theta'^2}}{\overline{B^2}}$$

Supposons un rapport (signal + bruit)/bruit de l'ordre de 40 décibels. Alors, on aura $C \# 20\,m^2/\lambda^2$ bits/stéradian soit, pour $m = 1$ cm, $\lambda = 0{,}5\ \mu$ environ 8×10^9 bits/stéradian ou $8 . 10^7$ bits pour $l = 10$ cm et $\Sigma = 1$ cm^2. Cette quantité d'information est équivalente à celle que peut débiter par seconde un canal d'environ 6 Mcs avec un rapport (signal + bruit)/bruit de 50 décibels (ce canal pourrait transmettre plus d'un millier de conversations téléphoniques nettement intelligibles). Nous avons raisonné sur un rapport (signal + bruit)/bruit de 40 décibels ; cela signifie entre les écarts-types $\sigma_{\theta'}$ et σ_B un rapport de l'ordre de 100. Prenons deux points objets correspondant à un écart $\sigma_{\theta'}$ par rapport au fond moyen $\overline{\theta'}$ et angulairement distants de λ/m (donc juste séparés) ; on sait que le minimum au centre de la tache globale est inférieur aux deux maxima voisins d'environ $0{,}2\ \sigma_{\theta'}$, c'est encore 20 fois plus grand que σ_B et la notion habituelle de pouvoir séparateur a tout son sens. Mais pour un rapport (signal + bruit)/bruit de 20 décibels on aurait $\sigma_{\theta'}/\sigma_B \# 10$ et l'écart qui permet de séparer les deux taches de diffraction n'est plus que de l'ordre de 2 fois l'écart-type du bruit ; il y a déjà une probabilité de 5/100 pour qu'il soit inférieur au bruit et la notion de pouvoir séparateur commence à perdre de sa précision. C'est encore plus marqué pour de plus faibles valeurs du rapport signal/bruit : par exemple, pour un rapport signal/bruit voisin de 10 décibels l'écart permettant de séparer les taches n'est que de l'ordre des 7/10 de l'écart-type et la probabilité pour que le bruit soit supérieur à cet écart est de l'ordre de la moitié. Cet exemple montre ce qu'apporte de nouveau l'introduction du point de vue statistique et l'intervention d'un terme de bruit lié à la précision limite avec laquelle on peut apprécier des différences d'éclairement significatives. Il semble raisonnable de penser que l'étude complète d'une transmission d'information optique ne peut ignorer cet élément.

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Capacity of an optical channel in the presence of noise

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Summary. — *An optical instrument can be considered as a transmission channel. The capacity of the channel is evaluated by applying standard results of information theory. The instrument is assumed to be free from aberrations and colour is not taken into account.*

Sommaire. — *Un instrument d'optique peut être considéré comme un canal de transmission. La capacité de ce conduit est évaluée en appliquant les résultats classiques de la théorie de l'information.*

Zusammenfassung. — *Ein optisches Instrument kann als Ubertragungskanal im Sinne der Informationstheorie angesehen werden. Dann lassen sich die Ergebnisse dieser Theorie zur Berechnung der Leistungsfähigkeit der optischen Instrumente benutzen.*

Introduction. — Information theory has been mainly developed in the field of electric communications. Only in recent times a few workers have made attempts at applying the theory to optics [1, 2, 3, 4]. This can be done from different points of view. Some of them are of purely theoretical or academic interest, while some others may lead to practical results only in very particular cases.

It is difficult to anticipate at present whether or not information theory will be able to bring about in the long run any material improvement to applied optics. It seems however that the best way to reach a conclusion in the near or far future is to try and apply to optics the central problem of communication theory. In order to do this one should not indulge too much in dealing with coherent or semi-coherent illumination, frequency analysis and all straightforward translations of well-known radio communication results into optical terminology. Such topics have been perhaps a little overstressed in optics. The eye is not the ear and any interpretation of vision in terms of frequencies is really too far-fetched (1).

Now, what has been called above the central problem of communication theory may be stated as follows : Given an information source and a transmission channel to find the way to transmit the greatest possible amount of information in the least time. The solution of this problem depends mainly on two factors : 1) the statistics of the source and 2) the capacity of the channel.

(1) This does not mean, of course, that Fourier analysis cannot be a very useful or even necessary tool in some intermediate steps, as is well known at least since Lord Rayleigh. Frequency analysis can also be important whenever the channel includes a radio-transmission stage, as in television. However this paper will only be concerned with a purely optical channel.

The statistics of the source represents in optics a fascinating problem which will be dealt with in a forthcoming paper. The present paper will only be concerned with the capacity of an optical instrument when considered as a transmission channel.

It is customary to define the capacity of a channel as the maximum number of bits that the channel can transmit per unit time. However it seems advisable slightly to change this definition in optics. For most optical instruments the time of observation is to be considered as practically unlimited. It will therefore be reasonable to define the capacity of an optical instrument as the number of bits that the instrument can transmit per single image or as the greatest possible number of bits obtainable from an image formed by the instrument.

In order to make the argument very clear and to keep the number of free parameters as low as possible only the black and white case will be discussed in the present paper. Color vision may be included in a later refinement of the theory. Further the instrument will be considered to be free from aberrations.

However, contrary to what is done in some wrong applications of information theory, the consideration of noise cannot be dispensed with and is essential to the argument. Indeed it is well known that, according to a correct definition, the entropy of a set of continuous probabilities, as are those to be considered here, would turn out to be infinite [5]. It is therefore only the difference of two entropies which can have a real significance. This difference is made when noise is taken into account.

Degrees of Freedom of an Optical Image. — First the following problem will be solved : How many

numbers are necessary to specify completely an optical image ?

To begin with consider a perfect optical instrument, having a one-dimensional pupil of width a. The illumination I_p in the image of a point source will have the form.

$$I_p = \operatorname{sinc}^2 \alpha \frac{a}{\lambda} \tag{1}$$

where α represents the angular position coordinate and sinc x, following Woodward's notation stands for the function $(\sin \pi x)/\pi x$. If $I_g(\alpha)$ indicates the illumination of the image of an extended object which would result from purely geometrical optics and $I_w(\alpha)$ the illumination of the same image when wave optics is taken into account, the following relation will obviously hold

$$I_w(\alpha) = I_g(\alpha) * \operatorname{sinc}^2 \alpha \frac{a}{\lambda} \tag{2}$$

where P*Q indicates the convolution of P and Q.

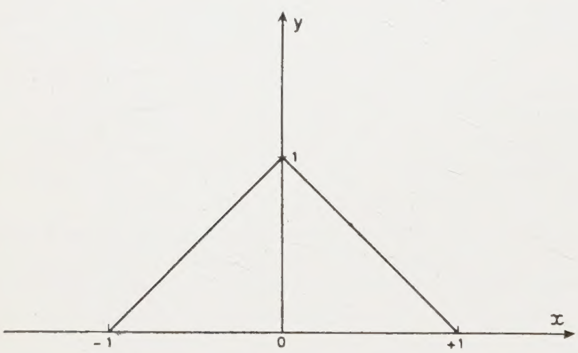

Fig. 1. — The function y = triang x.

According to a well-known theorem, the Fourier transform of the left side of (2) equals the product of the Fourier transforms of the two factors of the right side. Now it can be proved by standard methods that the transform of $\operatorname{sinc}^2 (\alpha a/\lambda)$ is given by (λ/a) triang $(f \lambda/a)$, where f stands for the frequency and the function y = triang x is that plotted in figure 1. As a result it can be written

$$i_w(f) = \frac{\lambda}{a} \, i_g(f) \operatorname{triang} f \frac{\lambda}{a} \tag{3}$$

where $i_w(f)$ and $i_g(f)$ represent the transforms of $I_w(\alpha)$ and $I_g(\alpha)$ respectively. It is evident from (3) and from figure 1 that $i_w(f)$ vanishes for $|f| > a/\lambda$. Thus $I_w(\alpha)$ contains no frequency exceeding a/λ. An application of the so-called sampling theorem [6] leads to the conclusion that $I_w(\alpha)$ is completely determined by giving its values at a series of discrete points (sampling points) spaced $\lambda/2\,a$ apart. If $\bar{\alpha}$ represents the total field angle of the instrument, the total number of sampling points will be $2\,\bar{\alpha}\,a/\lambda$.

Consider now a two-dimensional pupil, say a rectangle ab. The image of a point source will be represented by

$$I_p = \operatorname{sinc}^2 \alpha \frac{a}{\lambda} \; \operatorname{sinc}^2 \beta \frac{b}{\lambda}\,. \tag{4}$$

By a simple extension of the foregoing argument one would reach the conclusion that the number of sampling points which are necessary completely to specify the image I_w is given by $4\,\bar{\alpha}\,\bar{\beta}\,ab/\lambda^2$. If Ω designates the total solid angle of the field and S the pupil area this number can be written as

$$N = 4\,\Omega \frac{S}{\lambda^2} \tag{5}$$

One may say that N represents the number of degrees of freedom of the image. However, it should be noted that this number represents rather an order of magnitude than a precise figure. The sampling values cannot be chosen completely at will, because the illumination can never be negative. This question will be discussed in a forthcoming paper.

Entropy of the Image. — As a rule the illumination I at each sampling point will be comprised between 0 and a value I_{max} depending on the object. Maximum entropy will be obtained when all values of I inside the range $0 < I < I_{max}$ have equal probability. According to the usual definition, the entropy is then found to be equal to $\log I_{max}$.

The instrument has been assumed to be perfect from the geometrical standpoint. However, even in a perfect instrument, there is always some stray light. The distribution of stray light in the field may very well be irregular in some extreme cases. However, in the great majority of practical cases, stray light will be regularly diffused so as to form a uniform veil superimposed on the image. The corresponding illumination will be designated by I_d.

The resultant illumination at a general sampling point of the image will be termed the objective illumination and will be designated by I_0. The probability distribution for I_0 is uniform inside the range $I_d < I_0 < I_{max} + I_d$. The entropy of this distribution is still given by $\log I_{max}$.

Since the illuminations at different sampling points can be assumed to be independent of one another, the total entropy of the image will be given by $H = N \log I_{max}$, N being the number of degrees of freedom defined by (1).

It may be noted that the value of I_d will in most cases be proportional to the mean brightness of the object, or what amounts to the same, to the highlights of the object. To simplify matters it will be assumed that

$$I_d = p\, I_{max} \tag{6}$$

p being a numerical constant depending on the instrument. Thus the range of I_0 will be defined by $pI_{max} < I_0 < (1 + p)\, I_{max}$.

The Noise.— It has been remarked earlier that an expression of the entropy like that found in the preceding section is extremely conventional and has no meaning unless it is considered only as a first step in the evaluation of the channel capacity. A result of practical value can be obtained only when noise is taken into account.

Let x and y represent the transmitted and received signals respectively in the case of a general communication system. Then the mean transfer of information can be represented by the expression [5]

$$H(y) - Av_x\, H_x(y) \tag{7}$$

where $H(y)$ stands for the entropy of y, $H_x(y)$ for the same entropy when the transmitted signal x is known and Av_x indicates the average over all possible values of x. In making the difference (7) the infinite constants of two entropies cancel and the expression acquires a definite meaning.

It would seem quite natural to assume that in an optical channel noise be represented by stray light. However this is a little too theoretical, for in practice, apart from a few not very important cases, stray light does not show any randomness. The value of the entropy found in the previous section is independent of the stray light percentage.

In order to find out a reasonable cause of random noise to use in the theory somebody has taken into account such phenomena as photon fluctuation or fluctuations in the illumination of the object. Now it is evident that at the present stage, when a universally accepted theory of optical information is still to be developed, these refinements are better left out of the treatment. They may present some interest, if any, only in some particular cases to be investigated much later.

If one considers solely what happens in real or every day optics one cannot fail to reach the conclusion that the only cause of randomness and uncertainty which is worth mentioning is represented by the receptor. Optical noise [2] is mainly receptor noise. The portion of the channel preceding the receptor is practically noiseless.

In order to go further it is necessary to specify the receptor. It is quite natural to refer to the eye. On the other hand, since the capacity is defined as the maximum information obtainable from the image, the eye will be supposed to work in the best possible conditions. The brightness level and the magnification will be assumed to be sufficient for the resolving power to be limited by the instrument and not by the eye.

The conclusions which will be reached for the eye will also be valid for the photographic plate, provided that the grain be much finer than the resolving power.

It is well known that when a portion of the retina is illuminated with an objective illumination I_0, the subjective illumination I_s, i. e. the illumination perceived by the observer is in general somewhat different from I_0. There is an uncertainty in the observer's judgement, which is closely related to the differential threshold. This uncertainty represents noise and will be denoted by I_n. Thus one can write

$$I_s = I_o + I_n. \tag{8}$$

The probability distribution for I_n may be assumed to be the distribution holding in general for experimental uncertainties, i. e. the Gaussian one. If $p(I_n)\mathrm{d}I_n$ indicates the probability that the noise be comprised between I_n and $I_n + \mathrm{d}I_n$, one may write

$$p(I_n) = \frac{1}{I_N\sqrt{2\pi}} \exp\left(-\frac{I_n^2}{2I_N^2}\right), \tag{9}$$

I_N being the standard deviation.

On the other hand it is known from experiment that, inside the most favorable range of illuminations, I_N is practically proportional to I_0 (Weber's psychophysical law). Thus

$$I_N = \varepsilon\, I_0 \tag{10}$$

where ε is a numerical constant derived from experiment.

Capacity of the Optical Channel. — It is now possible to evaluate the capacity of the channel by means of the general expression (7).

The received signal y is, strictly speaking, represented by I_s. If I_s has a uniform probability, it is seen from (8) and (9) that the probability distribution for I_0 is not uniform. However this does not matter at all. Whatever the resultant distribution for I_0 may be, the capacity must be evaluated with that probability distribution for y which maximises $H(y)$. This would precisely be a uniform distribution, provided that I_s were limited to a well-determined range. The last condition is not strictly true in the case of I_s for both the upper and lower limits are slightly blurred by the psychophysical uncertainty. However it is clear that only a very small error will be committed if it is assumed that the two limits are fixed and still represented by pI_{max} and $(1+p)\, I_{max}$ respectively. In conclusion, for each degree of freedom $H(y)$ will be given by

$$H(y) = \log I_{max}. \tag{11}$$

The conditional entropy $H_x(y)$ is the entropy of I_s when I_0 is known. It is clear from (8) that $H_x(y)$ is simply the entropy of I_n, i. e. of the Gaussian distribution (9). Thus by recalling a standard result [6] it follows that

$$H_x(y) = \log\left(\sqrt{2\pi e}\; I_N\right). \tag{12}$$

Next the standard deviation I_N will be replaced by its expression (10) in terms of I_0 and the average will be made over all values of I_0. By carrying out the integration one obtains

(2) *Optical noise* is certainly a very odd expression. However a more appropriate one would hardly be likely to be widely accepted.

$$(13) \quad A v_x H_x(y) = \frac{1}{I_{\max}} \int_{pI_{\max}}^{(1+p)\, I_{\max}} \log\left(\sqrt{2\pi e}\; \varepsilon\, I_0\right) dI_0 = $$
$$= \log\left[\sqrt{\frac{2\pi}{e}}\; \varepsilon\; \frac{(1+p)^{1+p}}{p^p}\right] + \log I_{\max}.$$

Finally by inserting (11) and (13) into (7) the capacity of the channel per degree of freedom is found to be

$$(14) \quad C = \log\left[\sqrt{\frac{e}{2\pi}}\; \frac{1}{\varepsilon}\; \frac{p^p}{(1+p)^{1+p}}\right]$$

and by (5) the total capacity of the instrument is

$$(15) \quad \mathrm{C} = 4\,\Omega\, \frac{S}{\lambda^2} \log\left[\sqrt{\frac{e}{2\pi}}\; \frac{1}{\varepsilon}\; \frac{p^p}{(1+p)^{1+p}}\right].$$

As was to be expected from a sound theory, the value of $I_{\max}$ does not appear in the expression for the capacity. The capacity depends solely on the aperture and field of the instrument, on the differential threshold of the receptor and on the stray light percentage.

The above expressions for the capacity are of practical interest, for they lend themselves immediately to the derivation of numerical results. For instance putting $\varepsilon = .1$, as is reasonable to assume for the differential threshold of very close points and applying (14), one obtains for C the values which are plotted in figure 2 against the values of p. ; C is expressed in bits and p as a percentage.

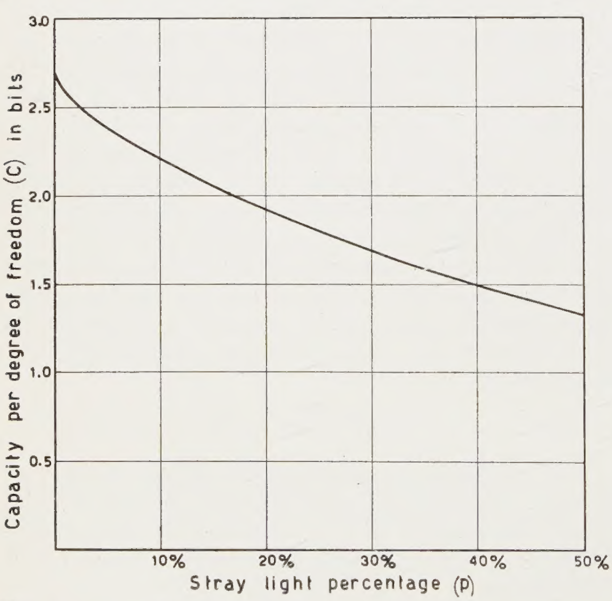

FIG. 2. — Capacity of an optical channel per degree of freedom as a function of stray light percentage.

If the value of ε is somewhat changed, the result, owing to the logarithm, does not change very rapidly. As a consequence it will never be very far from the truth to state that *a good optical instrument has a capacity of about* 2.5 *bits per degree of freedom.* Stray light reduces this capacity according to the universal law represented in figure 2.

Other numerical results which can be derived from (15) are the following : The human eye, when kept fixed, has a foveal capacity of about 10^5 bits. If the observer is allowed to turn his head in all possible directions the capacity becomes about 10^9 bits. A good telescope having a pupil of 100 mm diameter and 1° of field would have a capacity of about 10^8 bits.

Conclusion. — The author is well aware that not all of the assumptions made in the present paper have an absolute and general validity. However it is believed that a greater rigour, while terribly complicating the argument, would not alter the end results very much.

If information theory is to be of any use in optics and bring about some material progress, one must begin with a very simple and clear theory, discarding a great number of parameters wich are only of a secondary importance.

The theory developed above takes into account only the essential factors (excepting colour) of vision through a good optical instrument. The rule of thumb of 2.5 bits per degree of freedom is easy to remember. The dependence of capacity upon stray light is represented by the universal function plotted in figure 2. A change of the value adopted for the differential threshold merely brings about a shift along the ordinates.

All necessary elements are now at hand for investigating the redundancy of an optical system when employed to observe objects of given optical statistics. Starting from this point it will be interesting to try to eliminate the redundancy by suitably encoding the input data.

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A Proposal for a Radial Velocity Photometer

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SUMMARY. — *It is uneconomical, both in telescope time and in labour of reduction, to observe the details of a stellar spectrum if the sole object of the observation is to measure a radial velocity. An application of information theory by* WOODWARD *and* DAVIES *has suggested ways of markedly reducing this inefficiency by correlation methods of observation. Three principal procedures are proposed.*

(a) *The measurement of the position of a spectral line, recorded photographically by a microphotometer in which the transmission of the « slit » is made to agree with the expected shape of the line.*

(b) *The measurement of radial velocity by superposing a photograph of the spectrum to be measured on that of a star of similar type having a known radial velocity. The overall transparency of the superposed spectra is measured as a function of relative shift in the direction of the dispersion, and compared with the corresponding variation for superposed comparison spectra recorded on the plates. The method may succeed with a weak spectrum recorded by a relatively short exposure on a suitably pre-fogged plate.*

(c) *The direct measurement of radial velocity at the telescope by forming the spectrum of the star in a suitable spectrometer, into the focal plane of which can be placed a series of standard negatives of star spectra. The total transmission of the starlight through the appropriate negative is measured photoelectrically as a function of a displacement of the optical parts which simulates* DOPPLER *shift. The zero point of this shift is found in the same manner by superposing the spectrum of a comparison source on a comparison negative recorded alongside the standard star spectrum in the usual way.*

These methods, which give explicitly the probability distribution of radial velocity which results from the observations made, are shown to be in a certain sense ideal. A brief discussion is given of sensitivity and possible applications, particularly of method (c), *for which the name radial velocity photometer is suggested. The theory may also have applications to astrometric measurements.*

SOMMAIRE. — *Il n'est pas économique, tant en ce qui concerne le temps d'utilisation du télescope que le temps d'interprétation, d'observer les détails d'un spectre stellaire si l'unique objet de l'observation est la mesure d'une vitesse radiale.*

Une application de la théorie de l'information de WOODWARD *et* DAVIES *a suggéré des moyens d'améliorer singulièrement l'efficacité en utilisant des méthodes de corrélation.*

Trois procédés ont été proposés :

a) *mesure de la position d'une raie spectrale, enregistrée photographiquement, à l'aide d'un microphotomètre dans lequel la transmission de la « fente » est réglée de façon à s'accorder avec la forme prévue pour cette raie.*

b) *mesure de la vitesse radiale par superposition d'une photographie du spectre à mesurer et du spectre d'une étoile de même type dont la vitesse radiale est bien connue.*

La transmission d'ensemble des deux spectres superposés est mesurée en fonction de leur position relative dans la direction de la dispersion et comparée à la variation correspondante obtenue à l'aide de spectres de comparaison enregistrés sur les deux plaques.

La méthode peut réussir avec un spectre faible, enregistré avec une exposition relativement courte sur une plaque convenablement voilée au préalable.

c) *la mesure directe de la vitesse radiale au télescope en formant le spectre de l'étoile dans un spectromètre convenable dans le plan focal duquel peuvent être placées des séries de négatifs standards de spectres stellaires.*

La transmission totale de la lumière dispersée d'une étoile à travers un négatif approprié est mesurée photoélectriquement en fonction du déplacement des pièces optiques qui simulent le déplacement DOPPLER.

Le zéro de cette méthode est défini de la même manière en superposant le spectre d'une source de comparaison sur un négatif de référence enregistré parallèlement au spectre de l'étoile standard comme il est fait couramment.

On montre que ces méthodes qui donnent explicitement la distribution probable de la vitesse radiale résultant des observations, sont, à certains points de vue, idéales.

Une brève discussion est donnée sur la sensibilité, les applications possibles, particulièrement pour la méthode (c) *pour laquelle le nom de « photomètre pour la vitesse radiale » est proposé. La théorie peut avoir aussi des applications aux mesures d'astronomie de position.*

ZUSAMMENFASSUNG. — *Man wendet für die Beobachtung der Einzelheiten eines Sternspektrums unverhältnismässig viel Zeit am Fernrohr und Auswertearbeit auf, wenn der einzige Zweck der Beobachtung in der Ermittlung der Radial-Geschwindigkeit liegt. Auf Grund der Informationstheorie von* WOODWARD *und* DAVIES *kann man Korrelationsmethoden entwickeln, die diesen Aufwand erheblich herabsetzen. Dafür stehen drei Wege offen :*

a) *Man misst mit einem Mikrophotometer die Lage einer Spektrallinie auf der photographischen Platte, wobei die Schlitzbreite der Linie angepasst ist.*

b) *Man misst die Radial-Geschwindigkeit, indem man das zu messende Spektrum und das eines Sternes ähnlichen Typs mit bekannter Radial-Geschwindigkeit überlagert. Die allgemeine Durchlässigkeit der aufeinandergelegten Platten wird als Funktion der relativen Verschiebung in der Dispersionsrichtung gemessen und mit den Änderungen für die übereinandergelegten Vergleichsspektren verglichen. Die Methode ist auch bei schwachen Spektren anwendbar, wie sie bei kurzer Belichtungszeit auf einer vorbelichteten Platte erhalten werden können.*

c) *Man kann auch die Radial-Geschwindigkeit unmittelbar am Fernrohr messen. Dazu schickt man das Spektrum des Sternes in ein geeignetes Spektrometer, in dessen Brennebene eine Reihe Standardnegative mit Sternspektren aufgestellt werden können. Der gesamte Durchtritt des Lichtes durch das entsprechende Negativ wird photoelektrisch gemessen als eine Funktion der Kompensatorstellung, die die gleiche* DOPPLER-*Verschiebung hervorruft. Der Nullpunkt dieser Verschiebung wird in der gleichen Art festgestellt, durch Über Lagerung des Spektrums einer Vergleichslichtquelle auf einem Vergleichsnegativ, das neben dem Standardspektrum in der üblichen Weise abgebildet wird.*

Diese Methoden, die die Wahrscheinlichkeitsverteilung der Radial-Geschwindigkeit aus den Messungen ergeben, sind beinahe ideal. Es folgt eine kurze Diskussion der Empfindlichkeit, der Anwendungsmöglichkeiten, hauptsächlich für die Methode (c), *für die der Name « Radial-Geschwindigkeits-Photometer » vorgeschlagen wird. Die Theorie bietet auch Anwendungsmöglichkeiten bei astronometrischen Messungen.*

A photograph of the spectrum of a star provides a large amount of detailed information. When the object of the observation is to determine a radial velocity it is evidently uneconomical to base the determination on explicit data of high dimensionality (in the sense of requiring for its description a large number of parameters) much of which, moreover, was already known before the observation was made. The quantum

structure of light, together with the lack of perfection of radiation detectors, imposes a limit [1] on the rate at which information can be conveyed by means of a light beam of given intensity ; a star of zero magnitude provides about 10^6 quanta $\sec^{-1}$ cm^{-2} at ground level, and measurements made with a 24 inch telescope on a star of $12.^{m}5$ therefore depend on < 30,000 approximately random events per second, of which a photocell may record some 3,000 and a photographic emulsion or the human eye some 300, which are capable of representing a single parameter to two significant figures. This limitation to channel capacity provides a sufficient reason why photography of the details of a star spectrum is necessarily slow.

An application of information theory by Woodward and Davies [2] [3] [4] [5] has suggested a new method of measuring radial velocities whereby the informational inefficiency of the spectrographic method is much reduced [6]. This is achieved by making use of what is known in advance about the details of the spectrum, and by restricting the data obtained to quantities which as far as possible relate to the radial velocity alone. We shall see that the method gives promise of freedom from systematic errors and of giving measures in a few tens of seconds with a faint-limit similar to that for photo-electric photometry with the same observation time. The name « radial-velocity photometer » is intended to suggest the fact that the radial velocity is obtained explicitly (apart from trivial reductions) at the telescope by a photoelectric method. Briefly, the method consists in projecting a spectrum of the starlight onto a suitable mask, and measuring the amount of light transmitted through it. The variation of transparency along the mask, which may be made photographically, is related to the expected spectrum of the star in such a way that the individual lines of the spectrum (which may each contain far too little light to be observable in the time of observation) make additive contributions to the change in the total transmission when the dispersing system is given a small perturbation which simulates a Doppler shift.

1. **On a Theory of Observation.** — Suppose that an observed function $y(t)$ of an independent variable t is known to be the sum of noise fluctuations $n(t)$ and a function (« pulse ») $u(t - \tau)$ of known shape u but unknown shift of origin (« delay »)τ, and that it is required to estimate τ,

$$(1.01) \qquad y(t) = u(t - \tau) + n(t).$$

Evidently τ cannot be determined exactly, as this would require an infinite gain of information, and we therefore seek to find the probability distribution $p_y(\tau)$ which follows from the observation y. The product law for probabilities is in this case

$$(1.02) \qquad p(\tau, y) = p_0(\tau)\ p_\tau(y) = p_0(y)\ p_y(\tau)$$

where $p(\tau,y)$ is the probability of occurrence of the observation y and the delay τ, $p_0(\tau)$ the probability of τ prior to the observation, and $p_\tau(y)$ the probability of observing y when the delay is τ. Likewise $p_0(y)$ is the prior probability of y, and $p_y(\tau)$, which we wish to determine, is the probability of τ when y is observed. Expressed in the form of the law of inverse probabilities, 1.02 becomes

$$(1.03) \qquad p_y(\tau) = \mathrm{k}\, p_0(\tau)\ p_\tau(y)$$

where k is a normalising factor such that

$$\int_{-\infty}^{\infty} p_y(\tau)\ \mathrm{d}\tau = 1\ ;$$

thus k takes different values according to the equation in which it occurs.

In what follows, it can usually be assumed that the prior probability $p_0(\tau)$ is uniform in a certain range of τ, and zero elsewhere, so that in this range

$$(1.04) \qquad p_y(\tau) = \mathrm{k}\, p_\tau(y).$$

It is sometimes convenient to call $p_0(y)$ the « likelihood » of observing y on the hypothesis τ.

By 1.01,

$$(1.05) \qquad n_1 = n(t_1) = y(t_1) - u(t_1 - \tau).$$

If the noise n is Gaussian, with mean square value $\overline{n^2} = \mathrm{N}$, then the likelihood of $n(t_1)$ occurring is therefore

$$(1.06) \quad p_\tau(n_1) = \mathrm{k} \exp \left\{ -[y(t_1) - u(t_1 - \tau)]^2 / 2\,\mathrm{N} \right\}.$$

If the noise n is band-limited to $\pm$ w, and uniform within this range (i. e. the modulus of t its Fourier transform is statistically uniform in — w to w and zero outside this interval) then Woodward and Davies use the sampling theorem [7] to show that the likelihood of the whole waveform n in the interval $-\mathrm{T}/2 \leqslant t < \mathrm{T}/2$ is the product of 2wT independent terms of the form 1.06. By an inverse application of the same theorem

$$(1.07)\ p_\tau(n) = p_\tau(y) = \mathrm{k} \exp \left\{ -\frac{1}{\mathrm{N}_0} \int_{\mathrm{T}} [y(t) - u(t - \tau)]^2 \mathrm{d}t \right\}$$

where $\mathrm{N}_0 = \mathrm{N/W}$ is the mean noise power per unit bandwidth. By expansion and absorption into k of the terms in u^2 and y^2, which are independent of τ if T is sufficiently large compared to the range of τ considered that " end effects " do not occur,

$$(1.08) \qquad p_\tau(y)\ \mathrm{k} \exp \left\{ q(\tau) \right\}$$

$$(1.09) \qquad q(\tau) = \frac{2}{\mathrm{N}_0} \int_{\mathrm{T}} y(t)\ u(t - \tau)\ \mathrm{d}t.$$

By 1.03

$$(1.10) \qquad p_y(\tau) = \mathrm{k}\ p_0(\tau) \exp \left\{ q(\tau) \right\}$$

and if 1.04 applies

$$(1.11) \qquad p_y(\tau) = \mathrm{k} \exp \left\{ q(\tau). \right\}$$

Equation 1.01 may be written formally

$$(1.12) \qquad y(t) = u\ (t - \tau_0) + n_0(t)$$

where τ_0 is the true value of the delay, and n_0 the corresponding true noise, both of which are of course

unknown to the observer. Equation 1.09 then becomes

$$(1.13) \qquad q(\tau) = g(\tau) + h(\tau)$$

$$(1.14) \qquad g(\tau) = \frac{2}{N_0} \int_T u(t - \tau_0)\, u(t - \tau)\, dt$$

$$(1.15) \qquad h(\tau) = \frac{2}{N_0} \int_T n_0(t)\, u(t - \tau)\, dt$$

where g is known as the " signal function ", and h the " noise function " ; apart from a shift τ_0 of origin, g is the auto-correlation function of u. This separation of q into two components is artificial if only one observation y is considered ; but it corresponds to the way in which the observer might compare the result of a single observation y with the mean of a statistical set of y, which must in any case be postulated in order to define $p(y)$.

2. **On Ideal Interpretive Processes.** — It may be presumed that a well designed experiment is intended to answer some particular questions, and that the raw data obtained in the experiment contain implicit information relevant to these questions. The process of interpretation or " reduction " of the data consists in making this information explicit. Evidently the process of interpreting data cannot add information; (if it appears to do so, this information must be unfounded) and it should not destroy relevant information. In the language of information theory, it should conserve the entropy of the relevant probability distributions which result from the observation. An interpretive process which gives these probabilities explicitly exactly fulfills these requirements, and is therefore in this sense ideal. Within its range of applicability, the method of " correlation reception " of Woodward and Davies, outlined in § 1, is an ideal process of this kind.

The application considered by these authors is radar ; $u(t - \tau)$ is a pulse of known shape transmitted to a distant target, reflected, and received, with delay τ corresponding to the range of the target, in the presence of noise. The theory is developed to show how the ambiguity and accuracy of the measurement of τ, and the information gain in this measurement, depend on the ratio $R = 2\ E/N_0$ of E, the integrated mean square of u, to N_0. The extension is also made to the case that although the shape of the pulse u is known, its amplitude or even existence is not known in advance ; it is found that the appropriate probability distributions still depend on y only through q, and that they cannot be obtained more simply than as integrals over the joint amplitude-delay distribution.

It will be appreciated that the operation 1.09 is equivalent to a suitable frequency filtering of the waveform y. It is in fact the filtering which maximises the ratio of the peak signal height to the root-mean-square noise. This form of optimum filtering was derived by North [8], and it corresponds to weighting each small frequency interval in proportion to the square of the signal-to-noise ratio in the interval.

What has been gained by the analysis of Woodward and Davies in the case of radar is not a new practical procedure (since the North optimum filter had already been widely used) but a new interpretation which enables more effective use to be made of the result of a previously known method of observation. As well as providing the theoretical superstructive concerning the properties of the measurement as a function of R, their analysis proves that correlation reception is an ideal method for the case considered, and that it yields explicit probability distributions. One application of this may be noted. Since the probabilities resulting from repeated observations must be multiplied, 1.10 shows that the *sum* $\Sigma\, q_i$ of the successive q_i obtained is sufficient to give the probability resulting from the whole series of observations ; and by 1.09 $\Sigma\, y_i$ is also sufficient. *Differences* between repeated measures of noisy waveforms " to see if the feature repeats " have often been given more significance than appears justified in view of this result ; whenever probabilities are obtained explicitly, the procedure for combining observations follows without ambiguity or arbitrariness.

The literature, perhaps especially the older literature of astronomy, is full of references to the supposed magical properties of human beings as components of an observation system. Of course, in many cases an observer's " subjective " estimate of probabilities is based on facts which are perhaps poorly formulated and which are too complex to be handled by simple apparatus, or possibly even by a general-purpose computer, although this use of such machines is increasing. (At present, computing machines do not play master games of chess ; on the other hand it was reported that a computing machine gave a prediction, of the voting in a recent presidential election, which was " obviously " wrong and which was fulfilled.) But whenever, as in correlation reception of radar signals, the relevant probabilities can be evaluated explicitly in the apparatus or by routine calculation, then, as Woodward says, " no observer, however human " can possibly extract more from the data.

3. **Application of the Theory to Radial Velocity Measurements.** — Let the transmission be $T(x)$ at a distance x along the direction of dispersion of a photograph of a star spectrum. Suppose that the photograph is scanned by a microphotometer with a " slit " having transmission $S\ (x - \varkappa)$, where $\varkappa$ is the " setting " of the microphotometer. The intensity measured will then be

$$(3.01) \qquad I(\varkappa) = \int_P T(x)\ S(x - \varkappa)\ dx$$

where the integral is over the domain P of x which corresponds to the photograph. If there is an unblended line of form $U(x - \varkappa_0)$ in any part of the spectrum then

$$(3.02) \qquad T(x) = U(x - \varkappa_0) + n_0(x)$$

where n_0 is the noise fluctuation in the transmission caused by photographic grain. This noise is very nearly Gaussian [9]. It will be seen that 3.01 and 3.02 have the same form as 1.09 and 1.08 if S is made equal to U. In this case the probability distribution of the position $\varkappa$ of the line is, on these assumptions,

$$(3.03)\quad p_1(\varkappa) = \mathrm{k}\; p_0(\varkappa) \exp \left\{ \frac{2}{\mathrm{N}_0} \int_{\mathrm{p}} T(x)\, U(x-\varkappa)\, \mathrm{d}x \right\}$$

$$(3.04)\quad = \mathrm{k}\; p_1(\varkappa) \exp \left\{ \frac{2}{\mathrm{N}_0} I(\varkappa) \right\}.$$

For a pair of blended lines, an obvious (but practically rather complicated) extension of the theory would lead to a two-dimensional joint probability distribution for the positions $\varkappa_1$, $\varkappa_2$ of the lines.

An extension of this method is to regard the whole of the spectrum as the quantity U analogous to the pulse u in § 1. The microphotometer " slit " S then becomes a second photograph of the spectrum of a star, of the same type as the one investigated and of known radial velocity. The procedure is to superpose the two photographs, illuminate them uniformly, and to measure (for example, with a photocell) the integrated transmission $I(\varkappa)$ as a function of the relative shift $\varkappa$ of the two photographic plates. Subject to reservations discussed later, the probability of the radial velocity difference is then given by 3.04, provided that the dispersion law of the spectrum is such that a change in radial velocity gives equal linear shift $\varkappa$ all along the spectrum. This condition will ordinarily be satisfied sufficiently well only over limited ranges of wave-number, which should be masked off and measured separately. An obvious correction may be needed for the change in total transmission caused by end effects. The zero of the radial velocity would be determined by performing the same operation on comparison spectra recorded on the two plates in the usual way.

Finally, let the spectrum of a star be formed in a spectrograph and have brightness

$$B(x) = U(x - \varkappa) + n_0(x)$$

in the same way as in 3.02, where $\varkappa$ again corresponds to radial velocity and x is a function of linear distance along the direction of dispersion, such that $\varkappa$ can be given this meaning with the same scaling factor at all wave-numbers. If a photographic plate (or other mask) is placed in the focal plane of the spectrograph, and has transmission U, of the same form as the spectrum B under investigation, then the integrated transmission $I(\varkappa)$ again has the form 3.01. This transmission can be measured photoelectrically and the radial velocity determined in a similar way (but not an identical one, as we shall see) to in the previous method. A shift in $\varkappa$ does not of course in general correspond to a simple displacement of the plate, but to a more complex perturbation of the optical parts. The zero is found by running a comparison-spectrum source and comparing it by the same procedure with a similar comparison spectrum recorded on the plate.

It is this last method for which the designation " radial-velocity photometer " is specifically proposed.

4. **Theoretical discussion.** — It will have been noticed that the practical methods proposed in § 3, although they were suggested by the theory of Woodward and Davies outlined in § 1, do not conform to this theory in all respects.

The complication of the microphotometric method for the radial velocities of individual lines caused by the presence of blends will not be pursued here, as the principles involved are clear. The conditions also differ from those assumed in the theory because the photographic mean noise-power is correlated with the density along the plate. Ignoring this correlation broadens the probability distribution which is derived (i. e. it leads to an observation of lower "weight") but causes no systematic error. The loss of weight is usually quite small, and could be practically avoided by a suitable modification of the slit function S, since near to $\varkappa = \varkappa_0$ this it to a first order equivalent to distorting the amplitude of U, and the differential coefficient of the transformation can be chosen to equalise the mean noise at all amplitudes.

The method of superposing two plates differs from the theory for the additional reason that there is noise in both plates. If the true radial-velocity parameters ($\varkappa_0$ in § 3) of the two plates are now written $\varkappa_{\mathrm{S}}$ and $\varkappa_{\mathrm{T}}$, and the corresponding true noise functions n_{S} and n_{T}, then T, S, I have the form

$$(4.01)\quad T(x) = U(x - \varkappa_{\mathrm{T}}) + n_{\mathrm{T}}(x)$$

$$(4.02)\quad S(x) = U(x - \varkappa_{\mathrm{S}}) + n_{\mathrm{S}}(x)$$

$$(4.03)\quad \frac{2}{\mathrm{N}_0} I(\varkappa) = g(\varkappa) + h(\varkappa) + i(\varkappa)$$

$$(4.06)\quad g(\varkappa) = \frac{2}{\mathrm{N}_0} \int U(x-\varkappa_{\mathrm{T}})\, U(x-\varkappa_{\mathrm{S}}-\varkappa)\, \mathrm{d}x$$

$$(4.07)\quad h(\varkappa) = \frac{2}{\mathrm{N}_0} \int n_{\mathrm{T}}(x)\, U(x - \varkappa_{\mathrm{S}} - \varkappa)\, \mathrm{d}x$$

$$(4.08)\quad i(\varkappa) = \frac{2}{\mathrm{N}_0} \int \left\{ U(x - \varkappa_{\mathrm{T}})\, n_{\mathrm{S}}(x - \varkappa) + n_{\mathrm{T}}(x)\, n_{\mathrm{S}}(x - \varkappa) \right\} \mathrm{d}x.$$

Here g and h are analogous to the signal-and noise-functions comprising q in § 1. The first term in i represents the interaction of the s-noise with the known spectral form U. If a number of plates T are measured, using the same S as standard, this term represents a systematic " error " in the derived probabilities. If the " signal-to-noise ratio " R (§ 1) is as large as it usually will be in practice, the principal effect of this error will be to change the apparent zero $\varkappa_{\mathrm{S}}$, by an amount similar (if S, T are similar phtographs) to the random uncertainty in the values of $\varkappa_{\mathrm{T}}$ derived, and hence probably smaller than the systematic errors inherent in the spectrographic method (§ 5). When R is large, the term in $n_{\mathrm{T}}\, n_{\mathrm{S}}$ in 4.08 can be ignored.

In the radial velocity photometer, the substitution of a "live" spectrum for the illuminated recording T makes no important difference to g, the signal-function component of the transmitted intensity $I(\varkappa)$. Noise due to variations of intensity along the spectrum B, arising from interference of light reaching the telescope by different paths (colour scintillation), would be filtered in accordance with 3.01 if it remained stationary during the observation ; but this condition will seldom apply and in any case this noise is probably unimportant for faint stars. For such stars, the principal source of noise may be thought of as a time-dependent speckling of B, as "seen" by the photo-cell, arising from the quantum nature of light. The cell gives a discrete response, the emission of a photoelectron, to some fraction ε of the light-quanta reaching it, and in a good circuit these photoelectrons contribute to the final measurement with only small inefficiency. The signal-to-noise relations are the same as they would be with a perfect detector if the brightness level were multiplied by ε_e ($0 \leqslant \varepsilon_e < 1$), where ε_e is suitably chosen to be a little smaller than ε, to allow for the small additional noise introduced in measuring the photoelectrons. Alternatively, for the same brightness level, the quantum noise may be thought of as having been multiplied by the same factor as if the PLANCK constant h had become h/ε_e. The resulting effective speckling of B is not, as "wave" noise would be, filtered at all by the presence of the mask S. Consequently the noise relations are quite different from those of the previous methods, and must be re-discussed.

There is no correlation between the quantum noise at successive moments, consequently the noise (considered as a function of time t, or of $\varkappa$ if the spectrometer is scanned at a uniform rate $\mathrm{d}\varkappa/\mathrm{d}t$) has the uniform spectrum of "shot" noise [10]. It is not band limited, but its power per unit bandwidth N_0 is finite, and relations of the kind 1.07 (which is formally independent of W in the limit $W \to \infty$) can be applied. The value of N_0 is proportional, by the ordinary "shot" noise formula, to $I(\varkappa)$. Since we now have

$$(4.09) \quad I(\varkappa) = \int U(x - \varkappa_B)\, U(x - \varkappa_S - \varkappa)\, \mathrm{d}x + \text{noise of uniform spectrum}$$

$$(4.10) \quad = C[U]_{(\varkappa + \varkappa_S - \varkappa_B)} + \text{noise of uniform spectrum}$$

where $\varkappa_B$ is the radial-velocity parameter of the star spectrum B, $C\ [U]$ becomes the "pulse" of known form which is received in the presence of "uniform" noise, and the "delay" $\varkappa_B - \varkappa_S$ of which is to be measured. The probability distribution for $\varkappa_B - \varkappa_S$, is therefore given by 1.08, 1.09 with the substitutions

$$(4.11) \quad y = I$$

$$(4.12) \quad u = C\ [U]$$

$$(4.13) \quad p_y(\tau) = p_I(\varkappa_B - \varkappa_S)$$

and the practical procedure is to perform the operation 1.09 on the output of the cell, which is proportional to I. This may be done to a sufficient approximation by electrical filtering ; the filter used should have an indicial response (output as a function of time when an impulse is applied to the input) proportional to $C\ [U]_{(-\varkappa)}$ [5]. In practice, quite loose approximations to the ideal filter lead to only acceptably small loss in accuracy of the measurement.

The probability distribution so found ignores the correlation between I and the strength of the noise. This could be allowed for ; but by 1.08 the effect is negligibly small over the region of $\varkappa$ corresponding to appreciable probabilities, provided the signal-to-noise ratio R is $\backsim 3$ or more as it must be if an unambiguous measurement can be made (see WOODWARD [5], Chapter 6).

Moreover, p is the probability which results from making the measurement I, and the substitutions 4.11-4.13 do not depend in any way on I having the particular form 3.01. In fact, the theory of § 1 has been used to show how, if I has been observed, the best use can be made of this measurement ; but the theory does not show that I was the best measurement that could have been made.

For a given equivalent quantum efficiency ε_e, the best measurement can be shown to be that of B itself, made by means of an image detector, since this measurement is complete in the sense that from it any function of B, in particular 3.01 with $B = T$, can be evaluated with as high accuracy as by any other method in the same time of observation. This conclusion may seem to contradict the remarks with which this paper begins. The paradox is resolved when it is remembered that R measures the ratio of the *total* signal power to the spectral density of the noise. When R has the moderate value which suffices for unambiguous measurement of the delay (see WOODWARD [5] Chapter 6, eq. 35) the signal-to-noise ratio from point to point along a spectrum of reasonable length may be very small ; too small to allow any details of the form B of the spectrum to be significantly evaluated (loc. cit. Chapter 5, figs. 11, 13).

For example, the measurement of the star spectrum B, for radial velocity determination, could be made photographically in an ordinary spectrograph. The emulsion would be given an optimum pre-fogging, and the exposure on the star need only be long enough to give very small changes in photographic density. A sufficient exposure would give a spectrum on the plate which would probably be too faint to be seen by eye, but which could nevertheless be detected and located by the two-plate method, using the "key" of the prior knowledge of the form of the star spectrum represented by S in 3.01.

The method proposed in the previous paragraph has the practical advantage that it uses a conventional spectrograph. The rule-of-thumb value of the equivalent quantum efficiency ε_e for a photocell is 0.1 ; for a

sensitive photographic emulsion it is about 0.01 at optimum exposure, and may be ten times smaller for astronomical exposures subject to reciprocity failure [11]. Consequently the photographic method is at a disadvantage compared with the radial-velocity photometer by a factor of about 10 to 100 in the observation time needed to make a given measurement on a particular star. Roughly speaking, its advantage is that, once the spectrum B has been measured for a given integration time, the whole of the function 3.01 can be evaluated; whereas, although the radial velocity photometer can give a similar estimate of one value $I(\varkappa_i)$ of the function in the same observation time, an equal amount of additional time is needed for each successive measurement of an ordinate of I. It follows that, for a given observation time, the relative sensitivity of the two methods depends on the ratio of the interval in which the radial velocity can be assumed to lie, before the observation, to the permissible error in the velocity finally measured. The radial-velocity photometer will have a more favourable faint-limit if this ratio does not exceed about 100, and as this is about the average ratio to be expected in practice, the difference in limiting magnitude will usually be unimportant.

Reasons for preferring the photo-electric method are discussed in § 5.

If the use of an image detector to measure B is excluded, it remains to determine the best form of the observed waveform I, or (which is equivalent, since I is at our disposal through our control of S) the best form of the transmission S of the mask. Although the theory of § 1 does not give any direct indication of this optimum, it is clear that the choice $S = U$ is along the right lines. Presumably S should be closely related to U, in order to obtain a strong cross-correlation function I. Since this case differs from the others in that the noise is not filtered by S, enhancement of the higher frequences in S (for example by differentiating several times, compare 4.18) may be expected to give benefits similar to those of " pre-emphasis " in communication engineering. This enhancement should stop short, however, of making I too oscillatory.

The problem is to maximise the sharpness of p_τ in 1.03 with the substitutions 4.11-4.13. If R is reasonably large, this means maximising the (negative) curvature, at the value of $\varkappa$ corresponding to the true radial velocity, of $C\,[I]$ relative to the noise. By the Wiener-Kintchine theorem, and using the moment-generating properties of Fourier transforms, this curvature is proportional to

$$(4.14) \qquad \mu_2 = \int_{-\infty}^{\infty} |\, F\,[I_0]\,|^2\, \nu^2\, d\nu$$

where μ_2 is the second moment of the power spectrum of I_0, $F[I_0]$ means the Fourier transform of I_0 with argument ν, and I_0 is the signal component of I. By the Fourier product theorem and 3.01 with the substitution $T = U$

$$(4.15) \qquad F\,[I_0] = F\,[U]\,F^*[S]$$

$$(4.16) \qquad \mu_2 = \int_{-\infty}^{\infty} |\,F[U]\,|^2\,|\,F[S]\,|^2\, \nu^2\, d\nu.$$

The conditions on S are $0 \leqslant S \leqslant 1$, or if " negative " intensities are provided for (as might be done with a reflecting mask and balanced photocells, or by a number of beguiling methods using modulation or interference) $-1 \leqslant S \leqslant 1$. The noise power in I depends on the observation time and is proportional, as already noted, to I. These conditions are too remote from 4.16 to make it seem worthwhile, in view of the practical problems of making suitable masks, to pursue a formal solution at the present time.

Accordingly, the condition will be made that I should be compact (in the sense of being small outside a limited range of $\varkappa$), since otherwise the interval of $\varkappa$ that must be scanned is unnecessarily increased by end-effects. This requires that the phases of $F\,[U]$, $F\,[S]$ should differ, for the value of $\varkappa$ corresponding to the radial-velocity " coincidence ", by a constant angle, for convenience preferably 0 or π. As an approximation to the optimum S, we may then consider instead of 4.16 the absolute value of the curvature of I_0 at " coincidence " (which point is now taken as the origin of $\varkappa$), since this is proportional under the conditions assumed to

$$(4.17) \qquad \mu_2' = \int_{-\infty}^{\infty} F[I_0]\, \nu^2\, d\mu = $$

$$= \pm \int_{-\infty}^{\infty} |\,F\,[U]\,|\;|\,F\,[S]\,|\, \nu^2\, d\nu.$$

Further, we maximise this curvature relative to the noise in I_0. These conditions are approximately fulfilled if, at " coincidence ", the contribution to the curvature of I_0 from each interval $d\varkappa$ is given a weight proportional to its signal-to-noise ratio, which is proportional to $(U'')^2/U$. Since for S uniform the weight of this contribution would be as U'', the required S distribution is

$$(4.18) \qquad S = \text{Constant} \times U''/U.$$

When negative values of S are excluded, S is taken to be zero where 4.18 gives $S \leqslant 0$. It is then best to choose the constant in 4.18 to make S correspond to a " negative " of U. This follows because the changes in I have the same magnitude as for a " positive ", but " coincidence " corresponds to a minimum of I, with a consequent reduction in noise level at this point. The signal-to-noise ratio is also improved by making the multiplying constant as large as possible, even at the expense of allowing " saturation ", by the condition $S \leqslant 1$, to disturb the relative weighting to a moderate degree.

The practical approximation to the conditions 4.18 and $0 \leqslant S < 1$ is to use as the mask a negative photograph of the star spectrum, adjusting the pro-

cessing to give high density between the spectral lines, some narrowing of the lines due to enhanced contrast, and only small density in the cores of moderately strong lines. Despite the approximation of the arguments presented, it seems unlikely that any other form of mask would give greatly improved performance. In particular, methods based on locating the central zero of an anti-symetrical cross-correlation I do not, as might at first have been supposed, have any special advantages.

5. **Astronomical Consequences**. — The microphotometer method for locating the position of an unblended line, and the two-plate method of determining radial velocities are ideal interpretive processes, in the sense previously defined, in so far as the assumptions of the theory are realised. However tempting the exercise of skilled judgement and the making of laborious reductions may be, they can lead to improved results only in so far as they are based on more accurate assumptions, for example the recognition of discrete defects in the emulsion. It is easy to see qualitatively that the measurement of radial-velocity in this way gives greater weight to the stronger and sharper features of the spectrum, and the theory shows the conditions under which this weighting is the best possible.

The two-plate method provides in principle an objective method of determining spectral type. If, in the theory of § 1, the pulse shape u is no longer assumed to be completely known, but to be a known function of some parameter a, then by essentially similar methods a joint probability distribution $p_y(\tau,a)$ can be found. If a represents spectral type, the plate to be measured is autocorrelated by total transmission with a series of standard plates at the same dispersion, representing different types, and the distribution of transmitted intensity (suitably normalised to allow for variations in density between the standard plates) gives the joint radial-velocity and spectral-type probability function. Similar remarks apply to the radial-velocity photometer, since it gives rise to a signal-function of similar form.

An interesting property of the method is that quite small deviations from the standard cross-correlation function I of 4.01 would be detectable. Thus if the standard I of best fit were subtracted, quite weak peaks due to secondary spectra of different radial velocity might be revealed. Rough calculations indicate that the secondary component of a spectroscopic binary would probably be shown by this method if it gave > 0.01 of the light of the primary, provided the separation in radial velocity (and preferably also in spectral type) was large enough. The limiting light-ratio might be expected to be greater for the radial-velocity photometer, provided the stars are not very faint, since a photo-cell is free from the early overloading which affects photographic measurements.

It is envisaged that for observation by means of the radial-velocity photometer, a suitable spectrometer would be built and used photographically to prepare a " library " of standard plates. A star to be measured would first be observed with a two-colour photometer, and its spectrum then correlated in the spectrometer with the standard plate of spectral type corresponding to its observed colour, and possibly also with standards of neighbouring type as a check. The zero of radial velocity of the standard plates could be found by observing stars for which the velocities are known by conventional methods ; or independently by calibrating the G type standard against the sun or planets, and using visual binaries to extend this calibration to other types.

Each library plate would carry a comparison spectrum, which would be compared with a local light source of the same kind during each observation. It is sufficient and desirable for this comparison source to be fainter than the source used intermittently in photographic work, and a small radio-frequency discharge tube may be suitable.

The continuous use of the comparison source during the actual observation should markedly reduce errors caused by flexure in the spectrometer. All troubles caused by shrinkage and distortion of photographic emulsions are of course avoided, and the linearity of the photo-cell removes a source of error which DANJON [12] has shown to exist in photographic work. If a line or star image is slightly asymmetrical, due for example either to off-axis coma or to coma caused by slight figuring errors or thermal or mechanical distortion of the optical parts, then the position of its centre of gravity changes under non-linear distortion of the intensity. The effect is particularly serious in spectrographic work, since the absorption lines of the star are shifted in the opposite direction to the emission lines of the comparison source. In order to exploit these potential advantages of the photoelectric method, it will probably be necessary to " scramble " the illumination of the entrance slit by means of a " channel " or " wave-guide " optical element; this can be tapered if it is desired to obtain magnification [13]. It would be interesting to see if the well known systematic differences in spectrographs between star and surface illumination could be removed in this way. It will be necessary to control carefully the dispersion of the spectrometer, probably by providing a small adjustment of the overall magnification of the star spectrum which it forms.

There are a number of possible ways of obtaining the necessary optical perturbation which simulates DOPPLER shift, particularly if an echelle can be used. For an ordinary reflection grating, the scale $d\varkappa/d\varphi$ ($\varphi =$ radial-velocity) can be kept constant to $\pm$ 2.5 % over a spectrum subtending 20° at the collimater, by a linked rotation of the grating and shift of the plate ; this error does not depend on the other parameters of the design. This is about the maximum angular range used on any of the present Palomar spectrographs ; for an angle of 28° the error would be $\pm$ 5 %. The errors in measurement which result will be an order of magnitude smaller.

The faint-limit will depend on the efficiency of the spectrometer. This can probably be made higher than is possible with the optical restraints imposed in conventional spectrographs by the properties of the emulsion, especially if image-slicers [14] (which are fully effective with photoelectric measurement) can be used : but against this must be set the additional loss at the mask. Although it is difficult to make a firm estimate until some observing experience is gained with the method, it seems likely that this limit will be similar to that for photo-electric photometry. A signal-to-noise ratio of a few times at the cell suffices for a measurement of radial-velocity shift to about the average line-width ; in comparison, a signal-to-noise ratio of at least 100 is usually required in photometry, with an optical loss, mostly at the colour filter, of about three to five times. The faint-limit is not worsened, unless it is by a change in the purely optical losses, by increasing the resolution of the spectrum. Indeed, during the time of observation there may be less than one effective quantum in each resolved element of the spectrum. Thus there is no need to change the dispersion for stars of differing magnitudes.

The application of the method to galaxies would be very attractive, but unfortunately it is more difficult for galaxies than for stars to infer the spectrum from the colour. In particular, a large fraction of the visual emission of a galaxy may be in a single bright line. Evidently, if a spectrum is peculiar there is no substitute for observing its detailed structure, and correlation methods are then inherently unsuitable. At the other end of the brightness range, the method appears well adapted to STRUVE's proposal [15] for the precise routine measurement of the radial velocities of the brighter stars ; and in the intermediate range of magnitudes, the economies of the method, both in telescope time and reduction, might make practicable a radial-velocity survey.

It will be noted that the time of observation depends on the range of $\varkappa$ that must be scanned, and therefore increases with the range of radial velocity in which the object measured can be assumed to lie before the observation is begun. This accords with the fact that a measurement of given accuracy corresponds to a greater amount of information the wider the prior range of the quantity measured.

Much of what has been said here of the correlation method applied to spectral measurements will apply *mutatis mutandis* to positional measurements of stars or planets.

Information theory has sometimes been criticised on the ground that, although it has confirmed the correctness of procedures which had already been worked out empirically, it has led to no new practical methods. This criticism would seem to be met by the proposals of the present paper. Doubtless some of the present estimates of the performance of the methods proposed will prove to be optimistic ; but without a certain amount of optimism new methods would never be tried, and possibly some advantages have not yet been noticed. A large amount of experimental work will be needed to evaluate properly the true value of the proposals.

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Ist eine Lichtbewegung stets umkehrbar ?

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SUMMARY. — *The reversibility of a light path has often been considered as a theory tenable without restriction.*

In fact however, the reversibility cannot always be guaranteed when absorption is considered. The transparency of the boundary between two absorbing media is different in the two opposite directions and is (for normal incidence)

$$\frac{D_{1,2}}{D_{2,1}} = \frac{1+\varkappa_1^2}{1+\varkappa_2^2}$$

where $\varkappa_1$ and $\varkappa_2$ are the indices of absorption of the media considered.

The change of phase of a ray of light crossing such a boundary is also different for the two directions, whereas the changes of phase for the reflected part exactly as for the transparent media, differ by π for the opposite directions.

In the same way the refraction of a ray of light falling obliquely on the boundary of two absorbing media is not reversible. For several metals (Ag, Au, Cu) there exists an angle of incidence at which the wave penetrating into the metal does not change its direction. If an homogeneous wave in the reverse direction from the interior of the metal is made to fall on the boundary, no angle of incidence exists at which the wave emerges into the air without being refracted. For other metals (Fe, Pt) emergence from the metal into air is possible without refraction but not in the reverse direction.

SOMMAIRE. — *La réversibilité d'un chemin lumineux a été souvent considérée comme un théorème valable sans restriction. En fait, cette réversibilité ne peut cependant, pas toujours, être garantie quand on considère l'absorption. La transparence d'une surface de séparation entre deux milieux absorbants différents est différente dans les deux directions opposées et est (incidence normale acceptée)*

$$\frac{D_{1,2}}{D_{2,1}} = \frac{1+\varkappa_1^2}{1+\varkappa_2^2}.$$

où $\varkappa_1$ et $\varkappa_2$ sont les indices d'absorption des milieux considérés.

Le saut de phase de la lumière traversant une telle surface de séparation est aussi différente pour les deux directions, cependant que les rotations de phase pour la partie réfléchie, exactement comme pour les milieux transparents, se différencient de π pour les directions opposées.

De même la réfraction d'un rayon lumineux tombant sous incidence oblique sur la surface de séparation de deux milieux absorbants n'est pas réversible. Pour plusieurs métaux (Ag, Au, Cu) il existe un angle d'incidence sous lequel l'onde pénétrant dans le métal ne modifie pas sa direction. Si on fait tomber sur la surface de séparation une onde homogène inversée à l'intérieur du métal, il n'existe pas d'angle d'incidence sous lequel l'onde émerge dans l'air sans être réfractée. Pour d'autres métaux (Fe, Pt) la sortie du métal dans l'air est possible sans réfraction, mais non dans la direction inverse.

ZUSAMMENFASSUNG. — *Die Umkehrbarkeit einer Lichtbewegung wird vielfach als Theorem von uneingeschränkter Gültigkeit angesehen. Tatsächlich ist diese Umkehrbarkeit aber dann nicht immer gewährleistet, wenn die Absorption berücksichtigt wird. Die Lichtdurchlässigkeit D einer Grenzfläche zweier verschieden absorbierender Medien ist in entgegengesetzten Richtungen eine verschiedene und zwar ist (senkrechte Inzidenz vorausgesetzt) $\frac{D_{1,2}}{D_{2,1}} = \frac{1+\varkappa_1^2}{1+\varkappa_2^2}$ wobei $\varkappa_1$ und $\varkappa_2$ die Absorptionsindizes der beiden Medien bedeuten. Auch die Phasendrehung des durch eine solche Grenzfläche hindurchtretenden Lichtes ist in entgegengesetzten Richtungen verschieden, während die Phasendrehungen für den reflektierten Anteil, genau wie bei durchsichtigen Medien, in entgegengesetzten Richtungen sich um π unterscheiden. Ebenso ist die Brechung eines Lichtstrahls, der unter* **schräger** *Inzidenz auf die Grenzfläche zweier verschieden absorbierender Medien (im einfachsten Fall z. B. auf die Grenzfläche zwischen einem Metall und Luft) auftrifft, nicht umkehrbar. Bei manchen Metallen (Ag, Au, Cu) gibt es einen Einfallswinkel, bei dem die ins Metall eindringende Welle ihre Richtung nicht ändert. Lässt man umgekehrt aus dem Metall her eine homogene Welle auf die Grenzfläche fallen, so existiert* **kein** *Einfallswinkel, bei dem die Welle in Luft unabgelenkt austritt. Bei anderen Metallen (Fe, Pt) wiederum ist ein ablenkungsfreier Lichtübertritt aus dem Metall in die Luft, nicht aber in umgekehrter Richtung möglich.*

In der Optik erhält man in den meisten Fällen bei Umkehr einer Lichtbewegung den gleichen Strahlverlauf und die gleiche Schwächung der Intensität an den verschiedenen Grenzflächen, die die Lichtbewegung trifft. Diese Erfahrung ist so unmittelbar einleuchtend, dass man zunächst geneigt ist, diese Umkehrbarkeit in allen Fällen als gegeben anzusehen. In Wahrheit gibt es aber eine Reihe von Ausnahmefällen, von denen einige im Folgenden studiert werden sollen. Dabei ist immer genau anzugeben, was unter einer Umkehrung der Lichtbewegung verstanden wird und welchen Bedingungen die Lichbewegung unterworfen sein soll, ob sich das Licht in durchsichtigen oder in absorbierenden Medien fortpflanzt, ob nur Brechung und Reflexion auftritt oder ob auch wesentlich Beugung ins Spiel kommt usw.

Unter den vielen Theoremen, die sich mit der Reziprozität der Lichtbewegung beschäftigen, sei zunächst eines herausgegriffen, das von H. v. HELMHOLTZ in seiner " *Physiologischen Optik* " [1] folgendermassen formuliert wurde :

" Ein Lichtstrahl gelange von einem Punkte A nach beliebig vielen Brechungen, Reflexionen usw. nach dem Punkte B. In A lege man durch seine Richtung zwei beliebige, aufeinander senkrechte Ebenen a_1 und a_2, nach welchen seine Schwingungen zerlegt gedacht werden. Zwei ebensolche Ebenen b_1 und b_2 werden durch den Strahl in B gelegt. Alsdann lässt sich folgendes beweisen : Wenn die Quantität I nach der Ebene a_1 polarisierten Lichtes von A in der Richtung des besprochenen Strahles ausgeht und davon die Quantität K nach der Ebene b_1 polarisierten Lichtes in B ankommt, so wird rückwärts, wenn die Quantität I nach b_1 polarisierten Lichtes von B ausgeht, dieselbe Quantität K nach a_1 polarisierten Lichtes in A ankommen. Soviel ich sehe, kann hierbei das Licht auf seinem Wege der einfachen und doppelten Brechung, Reflexion, Absorption, gewöhnlichen Dispersion und Diffraktion unterworfen sein, ohne dass das Gesetz seine Anwendberkeit verliert. "

HELMHOLTZ hält also die Umkehrbarkeit des Lichtweges für allgemein gewährleistet, bis auf eine einzige Ausnahme, die er an einer anderen Stelle seiner Untersuchung bespricht. Diese Ausnahme erleidet das Prinzip der Umkehrbarkeit, wenn das Licht auf seinem Weg ein Medium passiert, das unter der Wirkung eines äusseren Magnetfeldes infolge des optischen Faradayeffektes die Polarisationsebene des Lichtes dreht.

Da nämlich wegen der raumfesten Orientierung der Richtung des Magnetfeldes die Drehung bei Hin- und Rückgang in gleicher Grösse und Richtung (bezogen auf ein *raumfestes* Koordinatensystem) sich vollzieht, erfolgt die Drehung (bezogen auf ein *mit der Lichtrichtung* fest verbundenes Koordinatensystem) bei Hin- und Rückgang des Strahles in *verschiedener* Richtung. Diese Verhältnisse sind schon von RAYLEIGH in einem hübschen Versuch [2] demonstriert worden (Abb. 1). (1)

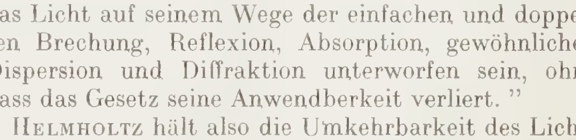

ABB. 1.

(1) Bemerkenswerterweise wird die Reziprozität nicht gestört, wenn man das magnetisch beaufschlagte und daher die Polarisationsebene drehende Medium durch eine optisch aktive Substanz, z. B. einen senkrecht zur optischen Achse geschnittenen Quarz ersetzt, da dann die Drehung der Polarisationsebene bezogen auf das mit dem Lichtstrahl verbundene Koordinatensystem stets im gleichen Sinne erfolgt.

Wenn ein Lichtstrahl durch zwei hintereinander gestellte Nicols geht, deren Schwingungsrichtungen einen Winkel von 45° miteinander bilden, so lässt sich auf diese Anordnung das Helmholtz'sche Theorem ohne weiteres anwenden. Schaltet man aber zwischen die beiden Polarisatoren eine mit Schwefelkohlenstoff gefüllte Röhre mit planen Glasfenstern, die sich in einem longitudinalen Magnetfeld (H : parallel zur Rohrachse) befindet, und sorgt man dafür, dass die Stärke des Magnetfeldes (dessen Kraftlinien z. B. von Nicol 1 zum Nicol 2 laufen mögen) so bemessen ist, dass die durch das Magnetfeld erzeugte Drehung der Polarisationsebene auch gerade 45° beträgt, dann wird in der einen Richtung das den ersten Polarisator passierende Licht ungeschwächt die nachfolgende Kombination von Magnetfeld und zweitem Polarisator passieren, während in der Gegenrichtung gar kein Licht durch die Anordnung dringen kann (Rayleighsche Lichtfalle).

Dass das Vorhandensein eines (raumfest orientierten) Magnetfeldes keine Reziprozität der Lichtbewegung zulässt, ist in den meisten Formulierungen von Reziprozitätstheoremen wohl beachtet worden. Weniger gründlich ist aber der Fall absorbierender Medien studiert worden, sodass sich hier einige Missverständnisse eingeschlichen haben, die zu beseitigen im folgenden unternommen werden soll.

In der Helmholtz'schen Formulierung wird die Ausbreitung einer Lichtwelle in absorbierenden Körpern ausdrücklich mit einbegriffen.

Das ist nun nicht immer zulässig. Zunächst seien nur die Intensitätsverhältnisse besprochen und zwar für den Spezialfall einer senkrecht auf die Grenzfläche eines absorbierenden und eines durchsichtigen Mediums auffallenden Welle. Auf die ausführliche Begründung wird an dieser Stelle verzichtet, da diese bereits früher [3] gegeben wurde. Es soll nur kurz das Ergebnis angegeben werden :

Fällt eine ebene Welle der Intensität 1 senkrecht von der Luftseite her auf eine Metallfläche, dann wird der Bruchteil ρ reflektiert und der Bruchteil δ ins Metall eindringen, derart, dass gilt. :

$$(1) \qquad \rho + \delta = 1$$

Fällt die gleiche Welle in der umgekehrten Richtung (also aus dem Metall kommend) auf die Grenzfläche, dann wird zwar wiederum der Bruchteil ρ reflektiert, in die Luft tritt aber ein anderer Bruchteil δ' aus, derart dass nicht mehr $\rho + \delta' = 1$ ist : $\rho + \delta' \neq 1$

Auf den ersten Blick scheint dieser Sachverhalt einen Verstoss gegen das Energieprinzip darzustellen. Tatsächlich ist dies aber nicht der Fall ; denn die über den Betrag δ überschiessende Energie $\delta' - \delta$ entsteht nicht aus dem Nichts, sondern rührt von dem gemischten Energieströmungsvektor her, der aus den Mischprodukten $E_r H_e$ und $E_e H_r$ (E_r, H_r und E_e, H_e : komplexe Amplitude der elektrischen und magnetischen Feldstärke der reflektierten bzw. einfallenden Welle) besteht, welch letztere in einem absorbierendem Medium nicht verschwinden. Man kann die Verhältnisse anschaulicher auch so interpretieren : In einer der Grenzfläche unmittelbar anliegenden Grenzschicht im Innern des absorbierenden Mediums ist bei Lichteinfall von der Seite des absorbierenden Mediums her die Absorption infolge der Ausbildung einer stehenden Welle geringer, als sie es nach dem Wert des Absorptionsindexes $\varkappa$ sein dürfte. Diese geringere Dämpfung ist der Grund dafür, dass mehr Energie vom Metall nach Luft übergeht als in der umgekehrten Richtung. Und zwar geht in Richtung : Metall $\rightarrow$ Luft um den Faktor $1 + \varkappa^2$ mehr Energie über als in Richtung Luft $\rightarrow$ Metall. Bedenkt man, dass $\varkappa$ bei Metallen grosse Werte annehmen kann, bei Silber z. B. 20, so ergibt sich die paradoxe Situation, dass beim Lichtaustritt von Silber nach Luft rund 20 mal soviel Intensität durch die Grenzfläche hindurchtritt als überhaupt auf sie auffällt. Das hat Konsequenzen für die Lichtdurchlässigkeit von Metallschichten. Besonders einfach werden die Verhältnisse bei " dicken " Metallschichten, d. h. bei solchen, die zwar noch wenige Prozent des auffallenden Lichtes hindurchlassen, bei denen aber keine merklichen " Zickzackreflexionen " mehr auftreten, sodass die Vorgänge an den beiden Grenzflächen als unabhängig voneinander angesehen werden können. In diesem Fall ist die Durchlässigkeit $\delta = \frac{I_d}{I_e}$ durch folgenden Ausdruck gegeben :

$$2) \qquad D = \delta_{1,2}\, e^{\frac{4\pi n \varkappa d}{\lambda}} \cdot \delta_{2,1}$$

Hierbei bedeutet $\delta_{1,2}$ den Schwächungsfaktor beim Übergang : Luft $\rightarrow$ Metall, $\delta_{2,1}$ den entsprechenden Faktor für den Übergang : Metall $\rightarrow$ Luft ; $e^{\frac{4\pi n \varkappa d}{\lambda}}$ bedeutet die Schwächung beim Durchsetzen der Schichtdicke d. $\delta_{1,2}$ ist gleich $1 - \rho$, wobei das Reflexionsvermögen ρ der Schicht wegen ihrer " Dicke " mit guter Näherung gleich dem Reflexionsvermögen des massiven Metalls gesetzt werden kann. $\delta_{2,1}$ ist aber nach dem bereits Gesagten — und hierin drückt sich eben die Irreziprozität des Lichtdurchgangs durch eine Grenzfläche zwischen verschieden absorbierenden Medien aus — nicht gleich $\delta_{1,2}$.

Fasst man die beiden Schwächungsfaktoren $\delta_{1,2}$ und $\delta_{2,1}$, die sich auf die beiden Grenzflächen beziehen, zu einem einzigen zusammen $\overline{\delta}$, so ergibt sich :

$$3) \qquad \overline{\delta} = (1 - I_r)^2 \, (1 + \varkappa^2)$$

oder indem man noch I_r durch die optischen Konstanten n und $\varkappa$ ausdrückt :

$$4) \qquad \overline{\delta} = \frac{16\, n^2\, (1 + \varkappa^2)}{[(n+1)^2 + n^2 \varkappa^2]^2}$$

An diesem Ausdruck [4] kann man leicht zeigen, dass $\overline{\delta}$ alle möglichen Werte zwischen 0 und 4 annehmen kann. Erstaunlich ist auf den ersten Blick, dass der

zusammengefasste Einfluss der beiden Grenzflächen nicht nur eine Schwächung ($\overline{\delta} < 1$), sondern auch eine " Verstärkung " der Lichtbewegung ($\overline{\delta} > 1$) bewirken kann. Physikalisch lässt sich leicht anschaulich machen, warum bei geeigneter Phasenlage dieser " Verstärkungsfaktor" optimal nur den Wert 4 annehmen kann. Der Faktor der Gesamtschwächung

$$D = \overline{\delta}.\mathbf{e}^{-\frac{4\pi n \varkappa d}{\lambda}}$$

ist natürlich immer kleiner als 1. Ebenso bleibt der Ausdruck D (trotz der Ungleichheit der Glieder $\delta_{1,2}$ und $\delta_{2,1}$) bei Richtungsumkehr unverändert und damit die Durchlässigkeit einer in Luft eingebetteten, absorbierenden Schicht (Metallschicht) in beiden Richtungen die gleiche.

Wenn nun auch die Verhältnisse bei einer Einfachschicht leicht überschaubar sind, so ist es doch nicht von vornherein sicher, dass bei einer geeignet aufgebauten Mehrfachschicht die Reziprozität der Intensitätsverhältnisse bei Richtungsumkehr erhalten bleibt.

Nun kann man zum mindesten für den Fall senkrechter Inzidenz und für eine Mehrfachschicht aus beliebig vielen, planparallelen Komponenten mit beliebigen optischen Konstanten n und $\varkappa$ durch direkte Durchrechnung zeigen [5], dass die Durchlässigkeit in beiden Richtungen die gleiche ist, vorausgesetzt, dass das Anfangs- und Endmedium die gleichen Absorptionsindizes besitzen, was im praktisch bedeutungsvollsten Spezialfall (Luft als Einbettungsmedium : $\varkappa = 0$) natürlich zutrifft. Einfacher noch als durch direkte Durchrechnung lässt sich dieser Nachweis führen, wenn man den Zusammenhang der Feldgrössen im Innern der Schicht in Form der Vierpolgleichungen anschreibt. Dann sind die Eingangsgrössen (magnetische bzw. elektrische Gesamtfeldstärke) mit den entsprechenden Ausgangsgrössen durch eine Kettenmatrix verknüpft, deren Determinante den Wert 1 hat. Bei Hintereinanderschaltung von n Schichten sind dann die Eingangsgrössen der ersten mit den Endgrössen der n. Schicht durch eine Matrix verbunden, die das Produkt aller Matrizen der einzelnen Schichten darstellt. Da im vorliegenden Fall die Determinante der Einzelmatrix den Wert 1 hat, hat nach dem Multiplikationstheorem der Matrizenrechnung auch die resultierende Kettenmatrix eine Determinante mit dem Wert 1, woraus sofort die Reziprozität der Durchlässigkeit folgt. Im einzelnen übersieht man auch, dass z. B. bei Einschaltung eines Magnetfeldes die Determinante der Verknüpfungsmatrix nicht mehr den Wert 1 behält, also auch nicht mehr die Reziprozität erhalten bleiben kann. Immerhin dürfte damit klar sein, dass, sofern man die Verwendung von Magnetfeldern ausser Betracht lässt und von der Hinzuziehung ganz neuer Effekte absieht, man durch keine Mehrfachschicht eine in entgegengesetzten Richtungen verschiedene Lichtdurchlässigkeit herstellen kann, ein Ziel, das aus technisch-praktischen Gründen erstrebenswert erscheinen mag und dessen Erreichung schon gelegentlich irrtümlicherweise behauptet wurde.

Das *Reflexionsvermögen* einer solchen Mehrfachschicht kann natürlich von beiden Seiten her ein verschiedenes sein. Man denke etwa an die im Handel befindlichen Sonnenbrillen, die auf der Aussenseite viel stärker reflektieren als auf der Innenseite und daher fälschlicherweise den Eindruck entstehen lassen, als ob die Gläser für den Brillenträger besser durchlässig wären als für denjenigen, der einen solchen Brillenträger betrachtet. Genau so verhält es sich mit den handelsüblichen Interferenzfiltern, die auf einer Seite metallisch reflektieren, auf der andern nicht. Eine Folge dieses seitenverschiedenen Reflexionsvermögens ist ein richtungsverschiedenes Absorptionsvermögen. Lässt man Licht von der stark spiegelnden Seite einfallen, so wird sehr viel weniger Licht absorbiert, nämlich der Betrag : $1 - \rho - \delta$, als wenn es von der schwach reflektierenden Seite einfällt : $1 - \rho' - \delta$. Im Sperrbereich des Filters, d. h. nahezu im ganzen sichtbaren Spektrum wirkt sich der Unterschied im Reflexionsvermögen voll aus, da hier die Durchlässigkeit praktisch Null ist. Es ist daher auch der Unterschied im Absorptionsvermögen für verschiedenen Lichteinfall noch grösser als im Durchlassbereich. Darum empfiehlt es sich bei der Benutzung der Filter diese mit der spiegelnden Fläche dem Licht entgegenzustellen, wenn man eine schädliche Erwärmung vermeiden will.

Die bisher besprochenen Beispiele waren immer nur Spezialfälle, wenn auch dem Fall der Mehrfachschicht wegen ihres beliebigen Aufbaus eine gewisse Allgemeingültigkeit zukam. Es wurde dafür aber nur mit senkrechtem Einfall gerechnet und nur Reflexion und Brechung, nicht aber Beugung in Betracht gezogen, da die einfallende Welle als unendlich ausgedehnt angesehen wurde.

Es erhebt sich nun die Frage, ob sich ein Reziprozitätssatz nicht auch unter ganz allgemeinen Voraussetzungen aussprechen und beweisen lässt.

In der Tat gibt es eine ganze Reihe von solchen Formulierungen unter recht allgemeinen Bedingungen. Da es zu weit führen würde, alle diese Theoreme einzeln aufzuführen und zu besprechen, soll nur ein einziges als besonders repräsentativ herausgegriffen werden und zwar der von SOMMERFELD und PFRANG ausgesprochene Satz [6]. Er behauptet die Vertauschbarkeit eines elektrischen Senders mit einem entfernt aufgestellten Empfänger, wobei wohl über die Art des Senders besondere Voraussetzungen gemacht werden (elektrischer und magnetischer Dipol), nicht aber über die besondere Art der Ausbreitung der elektromagnetischen Welle. Es kann also Reflexion, Brechung und Beugung ins Spiel kommen. Das Vorhandensein eines *Magnetfeldes* wird wieder ausgeschlossen. Der Sommerfeld'sche Satz baut auf einem sehr allgemeinen Theorem auf, das von H. A. LORENTZ [7] stammt und dessen Aussage sich prägnant in die folgende Formel fassen lässt :

$$5) \qquad \text{div}\ [E_2\, H_1] = \text{div}\ [E_1\, H_2]$$

Es bedeuten dabei E und H die elektrische bzw. magnetische Feldstärke zweier unabhängig voneinander verlaufender, elektromagnetischer Ausbreitungsvorgänge, wobei sich der Index 1 auf den einen, der Index 2 auf den andern Vorgang bezieht. Unter den Feldgrössen E und H sind komplexe Grössen zu verstehen, die die Phase mitenthalten. Die eckige Klammer bedeutet das Vektorprodukt. Wendet man auf die Gl. 5) den Gauss'schen Satz an, so kann man schliesslich schreiben :

$$6)\qquad \int_{K_1,K_2} [E_2H_1]\, d\sigma = \int_{K_1,K_2} [E_1H_2]\, d\sigma.$$

Die beiden Oberflächenintegrale sind über zwei Kugeln zu erstrecken, die die Ausgangsorte der beiden elektromagnetischen Wellen umschliessen. Diesen allgemeinen Satz wendet nun Sommerfeld auf einen elektrischen bzw. magnetischen Dipol an und kommt dabei, indem zunächst noch vorausgesetzt wird, dass die beiden Medien, die Sender und Empfänger umhüllen, keine Absorption zeigen, zu folgendem Ergebnis :

" Eine Antenne A_1 sende im Punkte O_1 und werde im Punkte O_2 von der beliebig gerichteten Antenne A_2 empfangen. Anderseits sende A_2 mit derselben Frequenz und Energie wie vorher A_1 und werde von A_1 empfangen. Dann ist die empfangene Feldstärke in A_1 dieselbe wie vorher die in A_2 und zwar unabhängig davon, wie das Zwischenmedium elektromagnetisch beschaffen und wie die Antennen geformt sind. "

Formelmässig lässt sich dieses Ergebnis schreiben :

$$7)\qquad \frac{n_1 \left| E_2^1 \right|^2}{n_2 \left| E_1^2 \right|^2} = \frac{n_1^2}{n_2^2}$$

Hierbei bedeuten n_1 und n_2 die Brechungsquotienten der Medien, die den Sender in der einen oder in der vertauschten Position umhüllen. E_1^2 und E_2^1 sind Feldstärken, die von dem in 1 aufgestellten Sender in 2, bzw. von dem in 2 aufgestellten Sender am Orte 1 erzeugt werden. $n_1 \left| E_2^1 \right|^2$ und $n_2 \left| E_1^2 \right|^2$ schliesslich sind die von dem Sender in den beiden Positionen am andern Ort erzeugten Intensitäten, die sich also bei gleicher Sendestärke des Senders in seinen beiden Stellungen wie die Quadrate der Brechungsquotienten verhalten (Gegenüber der Schreibweise bei Sommerfeld ist hierbei die Dielektrizitätskonstante ε durch n^2 ersetzt, was mit unserer optischen Fragestellung entschuldigt werden mag).

Im weiteren Verlauf zieht Sommerfeld auch absorbierende Medien mit in den Kreis der Betrachtung und gibt für sie folgendes Resultat an :

$$8)\qquad \frac{E_2^1}{E_1^2} = \frac{\left(\varepsilon_2 + i\,\frac{\sigma_2}{\omega}\right) v_2^{3/2} \sqrt{\varepsilon_1}}{\left(\varepsilon_1 + i\,\frac{\sigma_1}{\omega}\right) v_1^{3/2} \sqrt{\varepsilon_2}}$$

oder indem wieder optische Grössen eingeführt werden :

$$9)\qquad \frac{n_1 \left| E_2^1 \right|^2}{n_2 \left| E_1^2 \right|^2} = \frac{n_1^2 (1 + \varkappa_2^2)^2 (1 - \varkappa_1^2)}{n_2^2 (1 + \varkappa_1^2)^2 (1 - \varkappa_2^2)}$$

Wieder bedeutet bei den Feldgrössen der untere Index die Herkunft der Welle, der obere den Ort, an dem sie beobachtet wird. ε ist die Dielektrizitätskonstante, σ die Leitfähigkeit, ω die Kreisfrequenz, v die Fortpflanzungsgeschwindigkeit der Welle, n der Brechungsquotient, $\varkappa$ der Absorptionsindex. Man kann nun zeigen [8], dass sich die Aussage der Gleichung 9) nicht aufrechterhalten lässt, schon deshalb, weil der von Sommerfeld verwendete Hertzsche Ansatz für einen elektrischen Dipol in absorbierenden Medien zu Schwierigkeiten führt und sich mit seiner Hilfe nicht definieren lässt, was unter " gleicher Sendestärke " zu verstehen ist. Es ist daher zweckmässiger die Gleichheit der beiden Wellenvorgänge durch die Festsetzung zu fixieren, dass sich zwei Kugelwellen in den Entfernungen R_1 und R_2 von ihren Quellpunkten A_1 und A_2 zueinander verhalten wie :

$$10)\qquad \frac{n_1 \left| E_1 \right|^2}{n_2 \left| E_2 \right|^2} = \frac{e^{-\frac{4\pi n_1 \varkappa_1 R_1}{\lambda_0}}}{R_1^2} : \frac{e^{-\frac{4\pi n_2 \varkappa_2 R_2}{\lambda_0}}}{R_2^2}$$

wobei sich der Quellpunkt A_1 in einem Medium mit den optischen Konstanten n_1, $\varkappa_1$, der Punkt A_2 in einem solchen mit den optischen Konstanten n_2, $\varkappa_2$ befinden möge. Setzt man diese Bedingung an Stelle der Sommerfeldschen Bedingung " gleicher Senderstärke " ein, so ergibt sich unter sinngemässer Anwendung der Sommerfeldschen Beweisführung, dass sich die von der Welle 2 am Orte A_1 zu der von der Welle 1 am Orte A_2 erzeugten Intensität verhält wie :

$$11)\qquad \frac{n_1 \left| E_2^1 \right|^2}{n_2 \left| E_1^2 \right|^2} = \frac{n_1^2 (1 + \varkappa_2^2)}{n_2^2 (1 + \varkappa_1^2)}$$

Zu dem gleichen Resultat kommt man durch eine rein strahlenoptische Überlegung. Der Faktor $\frac{1 + \varkappa_2^2}{1 + \varkappa_1^2}$ kommt wieder durch die Irreziprozität des Übergangs vom Medium 1 nach 2 bzw. 2 nach 1 zustande, der uns schon von Anfang dieser Betrachtung her bekannt ist. Nach Betrachtung der Intensitätsverhältnisse sollen jetzt kurz die *Phasenbeziehungen* auf ihr Verhalten bei Umkehr der Lichtbewegung betrachtet werden.

Der Einfachheit halber sei wieder der Übergang einer ebenen Welle bei senkrechter Inzidenz durch die ebene Grenzfläche zweier Medien betrachtet. Sind beide Medien durchsichtig, so ändert sich bekanntlich die Phase der durchgehenden Welle in keiner Richtung, während die reflektierte Welle bei Einfall von der Seite des weniger brechenden Mediums einen Phasensprung von π, bei Einfall von der anderen Seite einen solchen von O erleidet. Stossen zwei absorbierende Medien mit den komplexen Brechungsindizes $\boldsymbol{n}_1$ und $\boldsymbol{n}_2$ aneinander, dann sind die Verhältnisse etwas

weniger einfach. Die Grenzbedingungen lauten dann :

$$12)\quad \begin{cases} \mathcal{E}_e - \mathcal{E}_r = \mathcal{E}_d \\ \boldsymbol{n}_1\, \mathcal{E}_e + \boldsymbol{n}_1\, \mathcal{E}_r = \boldsymbol{n}_2\, \mathcal{E}_d \end{cases}$$

wobei $\mathcal{E}_e$ die elektrische Feldstärke der einfallenden, $\mathcal{E}_r$ die der reflektierten und $\mathcal{E}_d$ die der durchgehenden Welle bedeutet. Der Lichteinfall erfolgt von Medium 1 nach Medium 2. Dann ergeben sich folgende Beziehungen, die die Änderung der Amplitude bei Reflexion und bei Übergang in das 2. Medium beschreiben :

$$13\text{ a)}\quad \frac{\mathcal{E}_r}{\mathcal{E}_e} = \frac{|\mathcal{E}_r|}{|\mathcal{E}_e|}\,.\,e^{i\delta_r} = \frac{\boldsymbol{n}_2 - \boldsymbol{n}_1}{\boldsymbol{n}_2 + \boldsymbol{n}_1}$$

und

$$13\text{ b)}\quad \frac{\mathcal{E}_d}{\mathcal{E}_e} = \frac{|\mathcal{E}_d|}{|\mathcal{E}_e|}\,.\,e^{i\delta_d} = \frac{2\,\boldsymbol{n}_1}{\boldsymbol{n}_2 + \boldsymbol{n}_1}$$

Bei Umkehr der Lichtrichtung (kleine Buchstaben für die Feldstärken) erhält man durch Vertauschen der Indizes :

$$14\text{ a)}\quad \frac{e_r}{e_e} = \frac{|e_r|}{|e_e|}\, e^{i\delta_r'} = \frac{\boldsymbol{n}_1 - \boldsymbol{n}_2}{\boldsymbol{n}_2 + \boldsymbol{n}_1}$$

und

$$14\text{ b)}\quad \frac{e_d}{e_e} = \frac{|e_d|}{|e_e|}\, e^{i\delta_d'} = \frac{2\,\boldsymbol{n}_2}{\boldsymbol{n}_2 + \boldsymbol{n}_1}$$

Aus diesen Beziehungen ersieht man, dass der Phasenwinkel der Lichtreflexion bei Lichtumkehr sich um π ändert, genau wie bei durchsichtigen Medien :

$$15)\qquad \delta_r = \pi + \delta_r'$$

Anders ist es beim Durchgang durch die Grenzfläche. Die Phasenänderung ist in diesem Falle richtungsabhängig und von Null verschieden.

$$16)\qquad \delta_d \neq \delta_d'$$

Wenn aber Gl. 13 b mit $\boldsymbol{n}_2$, Gl. 14b $\boldsymbol{n}_1$ multipliziert wird, so erhält man folgende Gleichheit :

$$17)\quad \boldsymbol{n}_2 \frac{\mathcal{E}_d}{\mathcal{E}_e} = \frac{2\,\boldsymbol{n}_2\boldsymbol{n}_1}{\boldsymbol{n}_2 + \boldsymbol{n}_1} = \frac{\boldsymbol{n}_1 e_d}{e_e} = \frac{|\boldsymbol{n}_2|\,|\mathcal{E}_d|}{|\mathcal{E}_e|}\, e^{i(\delta_d - \delta_2)} = \frac{|\boldsymbol{n}_1|\,|e_d|}{|e_e|}\, e^{i(\delta_d' - \delta_1)}$$

Es ist also die um den Phasenwinkel $\delta_2(\boldsymbol{n}_2 = |\boldsymbol{n}_2|\,e^{-i\delta_2})$ verminderte Phasenänderung δ_d für den Durchgang von Medium 1 nach Medium 2 gleich der um $\delta_1(\boldsymbol{n}_1 = |n_1|\,e^{-i\delta_1})$ verminderten Phasenänderung δ_d' beim Übergang vom Medium 2 nach Medium 1. Denkt man ferner daran, dass $\boldsymbol{n}_2\, \mathcal{E}_d = \mathcal{H}_d$ und $\boldsymbol{n}_1 e_d = h_d$ ist, dann kann man auch sagen, dass die magnetische Feldstärke der durchdringenden Welle aus der elektrischen Feldstärke der einfallenden Welle in *beiden Lichtrichtungen* durch Multiplikation mit dem gleichen komplexen Faktor (gleiche Änderung des Phasenwinkels) hervorgeht.

Zum Schluss sei noch die *Ablenkung* eines Lichtstrahls beim Durchgang durch die Grenzfläche zwischen einem durchsichtigen und einem absorbierenden Medium hinsichtlich ihres reziproken Verhaltens untersucht. Bekanntlich entsteht bei schiefem Einfall des Lichtes von der Luft her im Metall eine inhomogene Welle, bei der die Ebenen gleicher Amplitude parallel zur Grenzfläche liegen und die Wellenflächen (Ebenen gleicher Phase) einen Winkel φ mit ihr bilden. Je stärker der Grad der Inhomogenität (Querdämpfung längs einer Phasenfläche) der Welle ist, umso verschiedener ist die Fortpflanzungsgeschwindigkeit der Welle im Metall von derjenigen bei senkrechtem Einfall (homogene Welle). Die Inhomogenität hängt vom Einfallswinkel in der Luft ab. Bei manchen Metallen $[\boldsymbol{n}_1 = n_1(1 - i\,\varkappa_1)]$ gibt es aber nun einen Einfallswinkel, bei dem die Fortpflanzungsgeschwindigkeit der homogenen Welle in Luft gleich derjenigen der inhomogenen Welle in Metall ist. In diesem Fall tritt also ein schief aus der Luft her auf das Metall treffender Strahl ungebrochen in das Metall ein, eine optische Situation, die vom gewohnten Standpunkt der Optik durchsichtiger Medien etwas merkwürdig erscheint. Die Bedingung für das Auftreten dieses singulären Einfallswinkels lautet [9] :

$$18)\quad \sin^2 i_0 = 1 - \frac{n_1^4\, \varkappa_1^2}{1 + n_1^2\, \varkappa_1^2 - n_1^2}$$

Im Rahmen unserer Fragestellung nach der Möglichkeit einer unveränderten Umkehr der Lichtbewegung bleibt nun zu untersuchen, ob bei Einfall einer homogenen Welle in der umgekehrten Richtung d. h. vom Metall her nach der Luftseite beim gleichen Einfallswinkel der ablenkungsfreie Durchtritt durch die Grenzfläche erfolgt (Abb. 2).

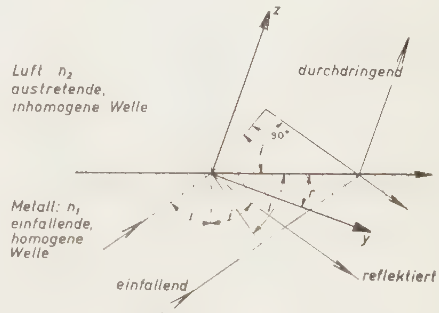

Abb. 2. — Das Koordinatensystem ist so gewählt, daß die Wellennormale der durchdringenden Welle gleich der positiven z-Richtung ist und das Einfallslot, z- und y- Achse in einer Ebene liegen.

Setzt man für die im Metall einfallende homogene Welle an : einfallende Welle : E senkrecht zur Einfallsebene :

$$19) \begin{cases} \mathcal{E}_{ex} = E_s . e^{i\omega\left[t - \frac{n_1}{c}(y \sin i' + z \cos i')\right]} \\ \mathcal{H}_{ey} = \boldsymbol{n}_1 E_s\, e^{i\omega[\ldots]} . \cos i' \\ \mathcal{H}_{ez} = -\boldsymbol{n}_1 E_s\, e^{i\omega[\ldots]} . \sin i' \end{cases}$$

wobei folgende Abkürzungen gelten mögen :

$$\boldsymbol{n}_1 = n_1 (1 - \mathbf{i}\,\varkappa)_1 \quad i' = i - r \quad i'' = i + r$$

dann erhält man für die reflektierte und die durchdringende Welle :

reflektiert :

$$20) \begin{cases} \mathcal{E}_{rx} = R_s\, e^{i\omega\left[t - \frac{n_1}{c}(y \sin i'' - z \cos i'')\right]} . e^{i\delta_{rs}} \\ \mathcal{H}_{ry} = -R_s\, e^{i\omega[\ldots]}\, e^{i\delta_{rs}} . \boldsymbol{n}_1 \cos i'' \\ \mathcal{H}_{zr} = -R_s\, e^{i\omega[\ldots]}\, e^{i\delta_{rs}} . \boldsymbol{n}_1 \sin i'' \end{cases}$$

durchdringend :

$$21) \begin{cases} \mathcal{E}_{dx} = D_s\, e^{by} . e^{i\omega\left[t - \frac{n_2}{c} z\right]} e^{i\delta_{ds}} \\ \mathcal{H}_{dy} = D_s\, e^{by} . e^{i\omega[\ldots]}\, e^{i\delta_{ds}} . n_2 \\ \mathcal{H}_{dz} = -D_s\, e^{by} . e^{i\omega[\ldots]}\, e^{i\delta_{ds}} . \frac{i c}{\omega}\, b \end{cases}$$

Die in Luft austretende Welle ist inhomogen und pflanzt sich mit einer anderen Geschwindigkeit fort als eine homogene Welle ! Man muss ihr daher einen besonderen Brechungsquotienten n_2 zuordnen. Für diesen gilt :

$$22) \qquad n_2^2 = 1 + \frac{b^2 c^2}{\omega^2}$$

indem der Brechungsquotient für die homogene Welle in Luft gleich 1 gesetzt wird, was angenähert zutrifft. b ist ein Mass für die Querdämpfung und ergibt sich aus den Grenzbedingungen zu :

$$23) \qquad b = -\frac{\omega}{c}\, n_1 \varkappa_1 \frac{\sin i}{\cos r}$$

Man kann dann das Brechungsgesetz in der Form schreiben :

$$24) \qquad \frac{\sin^2 i}{\sin^2 r} = \frac{1 + \frac{b^2 \lambda^2}{4 \pi^2}}{n_1^2}$$

Verlangt man jetzt Ablenkungsfreiheit : $i = r = i_0'$, so ergibt Gl. 23) $b = -\frac{\omega}{c}\, n_1 \varkappa_1 \operatorname{tg} i_0'$ und indem man diesen Ausdruck in Gl. 22 einsetzt

$$25) \qquad \frac{1 + \operatorname{tg}^2 i_0'\, n_1^2 \varkappa_1^2}{n_1^2} = 1$$

$$\text{oder } 26) \qquad \operatorname{tg}^2 i_0' = \frac{n_1^2 - 1}{n_1^2 \varkappa_1^2}$$

Bringt man die Bedingungsgleichung 18) in eine ähnliche Form, so folgt

$$27) \quad \operatorname{tg}^2 i_0 = -\frac{(n_1^2 - 1)(n_1^2 \varkappa_1^2 + 1)}{n_1^2 \varkappa_1^2 . n_1^2} = -\operatorname{tg}^2 i_0' \frac{1 \times n_1^2 \varkappa_1^2}{n_1^2}$$

Aus dieser Gleichung ist also abzulesen : Ein reeller Winkel i_0 schliesst einen reellen Winkel i_0' aus oder : Existiert bei schiefem Einfall einer homogenen Welle auf die Grenzfläche zwischen einem absorbierenden und einem nichtabsorbierenden Medium in der einen Richtung ein Einfallswinkel, bei dem die Wellennormale die Grenzfläche unabgelenkt durchdringt, so gibt es für die umgekehrte Richtung ausser senkrechtem Einfall keinen Winkel, bei dem die Welle die Grenzfläche unabgelenkt durchsetzt und umgekehrt. Gl. 26 und 27 lehren ausserdem, dass ein Winkel i_0 (Übergang Luft-Metall) nur existiert, wenn $n_1 < 1$, ein Winkel i_0' (Übergang Metall-Luft) nur, wenn $n_1 > 1$. Die gebotenen Beispiele sollten zeigen, dass man das Prinzip der Umkehrbarkeit der Lichtwege nicht kritiklos anwenden darf. Besondere Vorsicht ist am Platze, wenn es sich um absorbierende Medien handelt. Herrn Prof. Dr Cl. Schaefer und Herrn Prof. Dr Joh. Jaumann bin ich für wertvolle Diskussionen und die kritische Durchsicht des Manuskripts zu herzlichem Dank verbunden.

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Interferometric Methods for the study of diffraction images

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Summary. — *The photometric difficulties which are presented in direct measurements are discussed, and it is shown that a wave-front-shearing and a wavefront-tilting interferometer may be used to study respectively the transmission factors and point source diffraction images for any optical system.*

Sommaire. — *L'auteur étudie les difficultés rencontrées dans l'étude photométrique directe et montre que des interféromètres à dédoublement linéaire et angulaire peuvent être utilisés pour étudier respectivement les facteurs de transmission et les taches de diffraction pour un système optique quelconque.*

Zusammenfassung. — *Die photometrischen Schwierigkeiten der direkten Messung werden diskutiert. Dann wird gezeigt, wie ein Interferometer mit sich durchdringenden Wellenfronten und eines mit versetzten Wellenfronten benutzt werden kann, um entweder die Abbildungsgüte oder das Beugungsbild eines Sternes für ein optisches System zu untersuchen.*

1. **Introduction**. In the treatment of image formation from the standpoint of Fourier analysis, the transmission factor (response) for any spatial frequency in the object function is only easily found for coherent image formation, in which case the response factor is merely equal to the pupil function at an appropriate point. In other cases recourse to integrations is necessary. For example (Duffieux, [1]) has obtained the equivalent of the formula (using a different notation),

$$(1)\quad D(s,t) = \frac{1}{A}\iint_{-\infty}^{+\infty} f\left(x + \frac{s}{2}, y + \frac{t}{2}\right) f^*\left(x - \frac{s}{2}, y - \frac{t}{2}\right) \mathrm{d}x\,\mathrm{d}y$$

for the case of a self-luminous object, A being equal to $D(0,0)$; $f(x,y)$ is the pupil function, (x,y) being the pupil variables. The Fourier variables (s,t) may be written in terms of "number of lines" per unit length R, as is exemplified by the formulae :

$$(2)\qquad s = \frac{\lambda}{n \sin \alpha} R = \frac{\lambda}{n' \sin \alpha'} R'$$

where, primes denoting the image space, n is the refractive index and α is the angular inclination to the axis of rays from the circle $x^2 + y^2 = 1$ of the pupil. For a partially coherent object, it is necessary to define the object function by means of the complex transmission of the object plane as distinct from the intensity which has to be used for a self-luminous object. In this case, it has been shown (Hopkins, [3]) that a generalised transmission factor has to be defined involving two pairs of spatial frequencies. This transmission factor is shown to be given by the formula

$$(3)\quad C(m,n\,;p,q) = \frac{1}{A}\iint_{-\infty}^{+\infty} \gamma(x,y)\, f(x+m, y+n)\, f^*(x+p, y+q)\, \mathrm{d}x\mathrm{d}y$$

in which $\gamma(x,y)$ is the intensity in an effective source, and $A = C(0,0\,;\,0,0)$. For a coherent object $\gamma(x,y)$ reduces to a delta function, and for a self-luminous object $\gamma(x,y) = 1$ for all parts of the (x,y) plane. In this latter case grouping of the frequencies for which $m - p = 2\,s$ and $n - q = 2\,t$ leads back to the formula (1) (Hopkins, [3]). For a microscope using an annular stop in the outstage condenser, bounded by radii having numerical aperture σ_1, σ_2 times that of the objective, $\gamma(x,y) = 1$ inside the annulus defined by $x^2 + y^2 = \sigma_1^2$, $x^2 + y^2 = \sigma_2^2$, and is zero outside. The region over which the integration (3) has to extend is then the shaded area shown by way of example in figure 1(a). A simpler region results in the case of a self-luminous object, namely that shown shaded in figure 1(b). In each of these cases a circular aperture is assumed for the objective, and it may be noted that with a less simple shape very much more complicated regions of integrations may result. This is one reason, among others, why the analytical evaluation of these transmission factors presents mathematical difficulties.

If the optical system is free from aberration and defect of focus, $f(x,y) = 1$ for points interior to the pupil and is zero outside. Both (1) and (3) then merely call for the calculation of simple areas in the (x,y) plane, leading to results already well known. The response curves for pure defect of focus have been calculated analytically (see the accompanying paper " Transmission factors for incoherent objects in the presence of a defect of focus ") (1), and the effects of astigmatism are being studied by the same method (2). Beyond these cases the analytical difficulties become increasingly severe because of both the form of the integrand and the region of the integration, and it is for this reason that attention was turned to experimental methods for determining transmission factors.

A result of some importance may be derived by simple considerations relating to (3) and (1). Suppose that in evaluating (3), the pupil in question is circular

(1) Proceeding of Optical Conference, Florence 1954.
(2) Mr M. De is at present engaged on numerical calculations of the formulae he has obtained for this case.

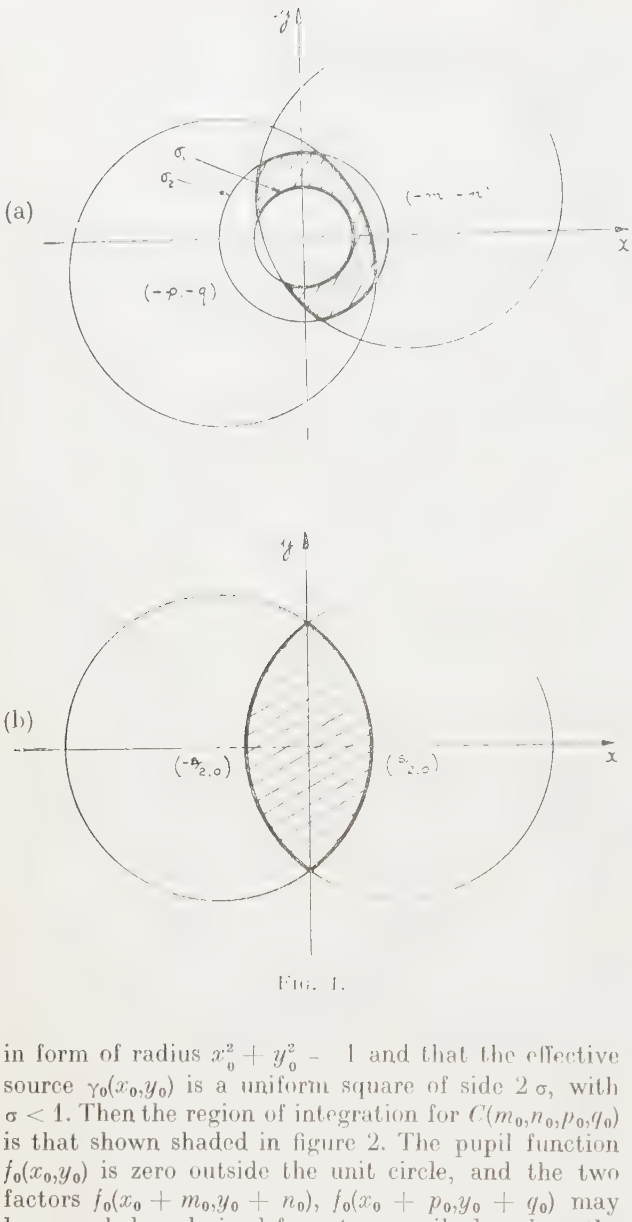

Fig. 1.

in form of radius $x_0^2 + y_0^2 = 1$ and that the effective source $\gamma_0(x_0,y_0)$ is a uniform square of side $2\,\sigma$, with $\sigma < 1$. Then the region of integration for $C(m_0,n_0,p_0,q_0)$ is that shown shaded in figure 2. The pupil function $f_0(x_0,y_0)$ is zero outside the unit circle, and the two factors $f_0(x_0 + m_0, y_0 + n_0)$, $f_0(x_0 + p_0, y_0 + q_0)$ may be regarded as derived from two pupils based on the points $(-\,m_0, -\,n_0)$, $(-\,p_0, -\,q_0)$ respectively. If we now adopt a rectangular coordinate system with x-axis along the line joining these two points and having the origin at the mid-point of that line, we obtain in place of

$$C(m_0, n_0, p_0, q_0) = \frac{1}{A}\iint_{-\infty}^{+\infty} \gamma_0(x_0, y_0)\, f_0(x_0 + m_0, y_0 + p_0) f^*(x_0 + p_0, y_0 + q_0)\, dx_0\, dy_0$$

the simpler expression

$$(4) \quad C(m, 0\,; -\,m, 0) = \frac{1}{A}\iint_{-\infty}^{+\infty} \gamma(x,y) f(x + m,y) f^*(x - m,y)\, dx\, dy$$

where $2\,m = \sqrt{(m_0 - p_0)^2 + (n_0 - q_0)^2}$. The function $f(x,y)$ is derived from $f_0(x_0,y_0)$ by a simple rotation of axes and $\gamma(x,y)$ is derived from $\gamma_0(x_0,y_0)$ by a similar rotation plus a translation. The physical interpretation of this result is that the transmission factor for any given two pairs of Fourier components, corresponding to a two dimensional object, is exactly the same as that for a certain frequency pair of a unidimensional object, provided the pupils are suitably oriented with respect to this object, and that the effective source is given an appropriate position and orientation. The converse of this means that it is possible to obtain all the image-forming properties of an optical system using only slits or grating-like line structures, no matter what the aberrations of the system. Moreover, the result is true for any shape of pupil and effective source. In the case of a self-luminous structure, $\gamma(x,y) = 1$ for the whole plane, so that it is only necessary in that case to rotate the axes in order to derive $f(x,y)$ from $f_0(x_0, y_0)$. In practice, of course, it would often be simpler merely to rotate the line structure relative to the pupil.

With the above result in mind, we may now confine attention to the transmission factors for line structures. Experimentally this implies the possibility of using scanning slits or slit objects in place of scanning holes or effectively point sources, with consequent photometric advantages.

2. **Direct scanning methods.** — In the case of a self-luminous object the transmission factors for different frequencies, that is the response function, of an optical system may in principle be determined by direct photometric measurements made on the image distribution of light. If we consider, for example, an incoherent line source of width small in comparison with the

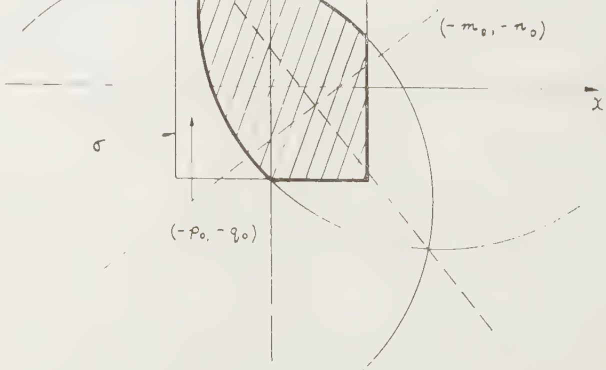

Fig. 2.

resolution limit, the intensity in the object may be described by

$$B(u) = \sqrt{2\pi}\,.\,\delta(u) \tag{5}$$

in which $\delta(u)$ is a delta function, and $\sqrt{2\pi}$ a simple normalising factor. The inverse transform of $B(u)$ is $b(s) = 1$, and the inverse transform of the image distribution of intensity is thus $b'(s) = D(s)b(s) = D(s)$, where $D(s)$ is the transmission factor. The image distribution is therefore given by

$$B'(u') = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{+\infty} D(s)\mathrm{e}^{\mathrm{i}u's}\,\mathrm{d}s \tag{6}$$

and, inverting this gives

$$D(s) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{+\infty} B'(u')\,\mathrm{e}^{-\mathrm{i}u's}\,\mathrm{d}u' \tag{7}$$

so that the intensity distribution in the image of a line source and the response function constitute a pair of Fourier transforms. Hence such an image, scanned by means of a very fine slit, would give the necessary information for the derivation of $D(s)$. In practice, however, one of the slits must be of width equal to not more than about one quarter of the resolution limit. For N. A. = 0.125 (that is an $F/4$ objective) the resolution limit is equal to 4λ so that a slit $\approx 1\lambda$ wide is needed. In such a case the amount of light available is necessarily small, apart from losses due to the partial polarisation known to exist with slits of this width. Of course, to study the sort of frequencies of interest in photography much wider slits could be employed, of the order of 20λ. But, even in this case, the objection remains that, of the small amount of light available only a small fraction is selected by the scanning slit. For an image distribution $B'(u') = (\sin u'/u')^2$, with $B'(0) = 1.00$, and a slit of width equal to $\pi/4$, the fraction of the total light accepted by the slit is only about 20 % even for the centre of the image. To measure a value near the maximum of the first fringe only about 1 % of the total light is employed. The central intensity $B'(0)$ would be much less than unity in the presence of even moderate amounts of aberration, and consequently the measurement of $B'(u')$ would more usually involve at most a few per cent of the total light. For these reasons it would seem that slit-scanning of the image of a line source is only practicable for the coarser frequencies, and even then is of limited accuracy.

Other photometric scanning methods are possible, however, and to consider these it is desirable to give a more general treatment of the problem. Let $B(u)$, $b(s)$ be the object function and its inverse transform respectively. Then the transform of the image function is

$$b'(s) = D(s)\;b(s). \tag{8}$$

Suppose now that the image is covered by a scanning screen, having a transparency $Z(u')$, and that the total amount of light transmitted is denoted by $T(u_0')$, when the scanning screen is displaced by a distance u_0'. In the case of a slit this would amount to a scanning function

$$Z(u') = 1,\ |\,u'\,| < a/2, \text{ and } Z(u') = 0,\ |\,u'\,| > a/2,$$

where a is the width of the slit, and u_0' would denote the coordinate of its centre. In scanning the image, $T(u_0')$ is measured as a function of u_0'. Now

$$T(u_0') = \int_{-\infty}^{+\infty} B'(u')Z(u' - u_0')\,\mathrm{d}u'$$

or, by the convolution theorem for Fourier transforms,

$$t(s) = b'(s)z(s). \tag{9}$$

Using (8), this leads to the expression

$$t(s) = D(s)\;b(s)z(s) \tag{10}$$

for the determination of $D(s)$. An important result follows from (10), namely that there is a reciprocal relation between the object $B(u)$ and the scanning function $Z(u')$. Hence we may either scan the image of an extended structure using a fine slit, or equally that of a narrow line source with an extended scanning screen, with identical results. The two cases are also photometrically of identical efficiency ; in the one case light from only part of the image of an extended source is ever utilised, whereas in the other case all of the light in the image of a line source may be utilised at least for one position of the scanning screen. This is true, for example, if the scanning screen consists of a " knife-edge ". In comparison with a slit-scanning of a line source, therefore, at least a five times increase in light to be measured is possible. In the presence of aberration a greater factor of improvement would result.

It is clear that the value of $D(s)$ cannot be obtained from (10) for any value of s for which either $b(s) = 0$ or $z(s) = 0$. Hence, if either the test-object or the scanning screen is periodic in form, only the values of $D(s)$ for those frequencies which are present in both the functions $B(u)$ and $Z(u')$ may be obtained. It follows that a set of test-objects, or alternatively of scanning screens, is necessary if one or the other of these is periodic in form. A special case arises when the test-object consists of an effectively line source, for which $b(s) = 1$, and a set of single-frequency scanning screens is employed. According to (7) the real and imaginary parts of $D(s)$ are given by the expressions

$$\frac{1}{\sqrt{2\pi}} \int_{-\infty}^{+\infty} B'(u')\cos u's\,\mathrm{d}u'$$

$$\frac{1}{\sqrt{2\pi}} \int_{-\infty}^{+\infty} B'(u')\sin u's\,\mathrm{d}u'$$

respectively, when the object consists of a line source. In the general case these two expressions give the real and imaginary parts of $D(s)b(s)$, where $b(s)$ is the

inverse Fourier transform of $B(u)$. If the scanning screen has the form

$$(11) \qquad Z(u') = \frac{1}{2}(1 + \cos su')$$

the total light transmitted by the screen is equal to

$$(12) \quad \frac{1}{2}\int_{-\infty}^{+\infty} B'(u')du' + \frac{1}{2}\int_{-\infty}^{+\infty} B'(u')\cos su'\, du'$$

when the screen is placed in a position where a point of maximum transmission coincides with the point where the principal ray of the geometric image intersects the plane of focus. If the scanning screen is displaced by half a period, the total light transmitted is given by a similar expression containing the imaginary part of $D(s)$. For the determination of $D(s)$ for limitingly small values of s, for which $D(s) \to 1$, one half of the total light is employed, no matter what the aberrations of the system. Moreover, the transmission factor is given directly, with a consequent gain in accuracy.

An objection inherent in the above method of determining $D(s)$, would seem to lie in the need to have available a range of scanning screens, each having a transmission accurately of the form (11). The practical difficulties involved in making such screens accurately are by no means trivial, especially when frequencies near the inherent limit of resolution are in question. For example, an objective of aperture $F/2$, used at 100 times magnification, would need screens as fine as 10 lines/mm for the limiting resolution. For a frequency of 50 lines/mm the scanning screen would need only 5 lines/cm with this magnification — but relatively fine screens would always be needed in testing systems designed to work at magnifications not very different from unity. However, given the necessary screens, 50 % of the total light is employed in the best case, and the photometric efficiency of the method is therefore directly related to the maximum permissible width of the line source.

The width of the source can be of the order of one quarter the length of period of the highest frequency to be measured, providing the variation of intensity across the source is known, for a correction may then be applied for the finite width. However, even for a frequency of 50 lines/mm a slit width of only 5μ is required, and a self-luminous source of this size having a regular and known distribution of intensity would not appear to be easily realised. The width of source would therefore need to be made not greater than about one tenth the length of period of the finest frequency, in order to behave nearly enough like an ideal line source. For the limiting resolution of any less the maximum width of source permissible would then be given by the expression a $= 0.05\ \lambda/(n \sin \alpha)$, in which n and $\sin \alpha$ refer to the object space containing the line source.

The interferometric method for the determination of $D(s)$ described below avoids the need for scanning screens, and appears to possess a significant photometric advantage over the direct scanning methods already discussed. Moreover, the new method is immediately applicable to partially coherent objects, using any method of observation such as dark-ground or phase-contrast. A similar technique, using wavefront-tilting in place of shearing, is found to have similar advantages for the experimental study of point-source diffraction images.

3. **A shearing interferometer for the determination of transmission factors.** — In figure 3 is shown the lens system L whose response function is to be measured ; for simplicity it is assumed that the system is intended for use on an infinitely distant object. From a point of the source S an aberrated wave-front W is produced. The waves reflected from and transmitted by the diagonal mirror D are reflected back along their incident directions by the two right-angle mirror pairs M, P — or better by two corner-cube mirror systems. By means of lateral displacements of M and P the reflected wave can be sheared off-axis without tilt.

Let W_m, W_p be the two wave-fronts sheared respectively by amounts $-m$, $-p$, where (m,p) is the frequency pair whose response factor is to be measured. The disturbances produced at the point (x,y) of the plane EE by the waves are respectively

$$(13) \quad \exp\left\{ ik\, W(x+m,y) \right\}, \quad \exp\left\{ ik\, W(x+p,y) \right\}$$

where $W(x,y)$ is wavefront aberration of the system L. Thus the intensity at the point (x,y) of the interference pattern in the plane EE is expressed by

$$\left| \exp\left\{ ik\, W(x+m,y) \right\} + \exp\left\{ ik\, W(x+p,y) \right\} \right|^2$$
$$= 2 + 2\cos k \left\{ W(x+m,y) - W(x+p,y) \right\}$$

and the total light flux in the interference pattern contained in a region A of the plane EE is given by the integral.

$$(14) \quad B = 2\iint_A dx\, dy + 2\iint_A \cos k \left\{ W(x+m,y) - W(x+p,y) \right\} dx\, dy.$$

If now the region A is made to coincide with the region common to the effective source and the two pupils centred on $(-m,0)$, $(-p,0)$ respectively, this region being isolated by means of a suitably shaped mask, the second term in (14) gives the real part of the response (transmission factor) $C(m,p)$ for the frequency pair (m,p) for the coherence conditions corresponding to the effective source employed. For incoherent image-formation the effective source covers the whole (x,y) plane, and in this case it requires a mask to limit the region A to that common to two pupils

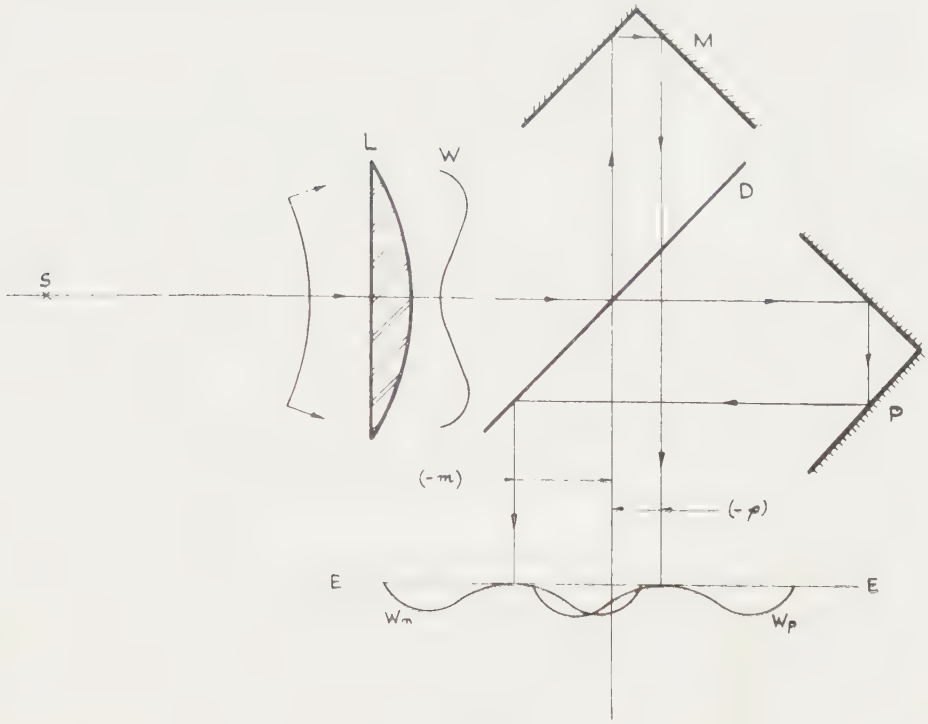

FIG. 3.

centred on $\left(\pm \frac{s}{2}, 0\right)$ for the component of the object intensity function of frequency s.

The total light flux in the fringe pattern is given by (14) when the path difference between the two arms of the interferometer is zero. If a path difference equal to $\lambda/2$ is now introduced, the fringe pattern changes and the total light flux is expressed by (14) but with the cosine there replaced by the sine. In this case the total light flux yields the imaginary part of the response factor $D(s)$.

Since only shear along one azimuth is needed, the light source S can be a line source. The effect of the finite width of the source is most easily appreciated by considering waves from a small length of the source situated on-axis. Each of the two wavefronts W_m, W_p deriving from the axial point of the source is sheared parallel to itself in its own plane, so that for an unaberrated wave no path difference is introduced. In contrast the two wavefronts from a point of the source at an angular distance β from the axis in the plane of the diagram, are separated by a distance βd when sheared an amount d. For frequencies near the limit of resolution d is equal to the diameter of the lens under test L, and $\beta_{max} = a/2$ where a is the width of the source and F the focal length of L. Hence the path difference is given by $a \sin \alpha$, for waves deriving from the extreme edges of the source. This error can clearly be halved by arranging a path difference of half this amount to exist between the wavefronts from the axial point of the source. The width of the source must therefore be such that $a \sin \alpha$ is only a smal fraction of the wavelength.

To arrive at a numerical value for this tolerance it is convenient to apply a result from the theory of partial coherence (HOPKINS, [2]). When W_m, W_p are given the maximum shear, equal to d the diameter of the lens L, we require to produce interference between wave elements which emerge from diametrically opposite points of the aperture of L. The whole of the lens aperture must therefore be illuminated coherently. For a circular source of angular radius β, the coherence is > 0.88 for a region of the illuminated plane of diameter

$$d = \frac{0.16\,\lambda}{n \sin \beta} . \tag{15}$$

For a line source of angular half-width β, the same formula holds, the coherence then having a lower limit of 0.84. Putting $\beta = \frac{a}{2F}$ as before, gives for the permissible width of source the expression

$$a = \frac{0.16\,\lambda}{n \sin \alpha}$$

which is three times that estimated for the most favourable direct scanning method. With a source of this size the associated error in the measured response factor is of the order of 2 % for a frequency equal to half the resolution limit, and increases to about 10 % for frequencies in the neighbourhood of the limit.

From what has been said above, it appears that the method proposed here has a significant photometric

advantage over direct scanning methods, and is also applicable up to the limit of resolution and under any conditions of coherence and observation. Moreover, no auxiliary scanning screens are required.

It is, of course, only the principle of the interference method which has been described here. Mr L. R. BAKER has evolved a practical technique for making the measurements and has designed and constructed a shearing interferometer embodying this principle. Mr BAKER has found it possible to make some preliminary measurements using only visual flicker photometry. He will publish a full account of the work at a later date.

4. **A wavefront tilting interferometer for the study of point-source diffraction images.** — The photometric difficulties presented in attempting to measure the distribution of light in the diffraction image of a point source are much greater than those encountered in the slit scanning of the image of a line source. For, in place of a line source and a scanning slit, a point source and a pinhole scanning aperture are required. On each count a hundredfold reduction in the amount of light to be measured can easily occur. The accuracy of direct scanning methods is therefore limited. Moreover, no information is obtained regarding the phase distribution in the pattern. These difficulties, it will be shown, can be avoided by the use of an indirect method depending on interference.

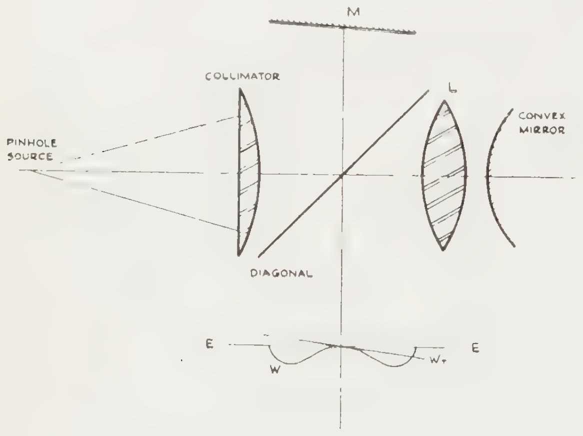

FIG. 4.

If the wavefront aberration of a lens system is denoted by $W(x, y)$, (x,y) being pupil variables, the complex amplitude at the point (u,v) of the Fraunhofer diffraction image of a point source is given, apart from a constant factor, by

$$(16)\quad F(u, v) = \iint \exp \mathrm{i}\left\{kW(x, y) + ux + vy\right\} \mathrm{d}x\,\mathrm{d}y$$

the integration extending over the region of the pupil. The image variables (u, v) have their usual significance.

Consider now a lens L under test with the TWYMAN-GREEN interferometer, as in figure 4 ; and, to simplify matters, assume that it is the diffraction image associated with the doubled aberration of L that is sought. denoting this doubled aberration by $W(x,y)$. To study the complex amplitude at the point (u,v) of the diffraction image, the mirror M is tilted so that the normal to the plane wave reflected from M is inclined at angles (β, γ) to the axes of x and y respectively, where $x \sin \beta = -u$, $x \sin \gamma = -v$, the refractive index being taken equal to unity. The disturbance at a point (x,y) of the plane EE now results from the interference between the aberrated wavefront W and the tilted wavefront W_T, giving an intensity equal to

$$\left| \exp\left\{\mathrm{i}k W(x,y)\right\} + \exp\left\{-\mathrm{i}\, ux + vy)\right\} \right|^2$$
$$= 2 + 2\cos\left\{k W(x, y) + ux + vy\right\}$$

apart from a constant factor. If, as before, a region A of the plane EE is masked off, the total light flux through A is expressed by

$$(17)\quad B = 2\iint_A \mathrm{d}x\,\mathrm{d}y + 2\iint_A \cos\left\{k W(x, y) + ux + vy\right\} \mathrm{d}x\,\mathrm{d}y$$

the second term of which gives the real part of the diffraction integral (16). This assumes equality of path length in the two arms of the interferometer. If a path difference of $\lambda/2$ is introduced, an expression similar to (17) is obtained for the total light flux in the new fringe system, but with the cosine in the integrand replaced by the sine. The second term of this expression gives the imaginary part of $F(u,v)$. Thus by measurements of the total light flux in these interference patterns, the amplitude and phase in the diffraction pattern is obtained.

We may note that, whatever the size of pinhole source, one half of the total light is used in measuring the real part of the complex amplitude at the central point of the image of an unaberrated wave. By contrast a pinhole aperture transmits only about 6 % of the total light even in this, which is the best case. At a point in the diffraction pattern having an amplitude equal to 0.10 times that at the centre of the pattern, the second term in (17) is only reduced by a factor 0.10, whereas the intensity at such a point is reduced by a factor 0.01. Hence, in this case, the interferometer is required to measure 5 % of the total light from the source, whereas the pinhole scanning aperture collects only 0.06 % of the light. Being phase sensitive, the interference photometer proposed here is capable of much greater accuracy than direct photometric scanning methods What is more the only restriction on the size of the pinhole, given a suitably designed system, is that it should be small enough to satisfy paraxial

optics. In contrast to this a pinhole of diameter not greater than about one quarter the resolution limit is essential in direct scanning of the image.

The use of the wavefront-tilting interferometer is not restricted to the study of point source diffraction images. It is not difficult to show that the transmission factors for defocused and astigmatic images may also be obtained with the instrument.

Mr M. De has designed an interferometer which uses the wavefront-tilting principle, and the instrument will soon be used for studies of the kind described.

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[2] Hopkins, H. H., *Proc. Roy. Soc.*, A, **208**, 1951.
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One radius doublets

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Summary. — *It is proved that with a suitable choice of glasses a one radius doublet has as good correction on the axis (for the object at infinity) as the best possible two component thin system. The ratio of the indices of refraction for crown and flint must be* 1 *to* 1.066. *Coma is not corrected, but small. The influence of thicknesses is discussed.*

Sommaire. — *Etude des propriétés de systèmes comportant une lentille de crown équiconvexe collée à une lentille de flint plan concave (système à une courbure).*
La correction sur l'axe (pour un objet à l'infini) est aussi bonne que celle des systèmes minces à deux éléments, à la condition que le rapport entre les indices de réfraction du crown et du flint soit de 1 *à* 1,066. *La coma n'est pas corrigée mais elle est petite. On tient compte de l'épaisseur.*

Zusammenfassung. — *Es werden Dubletts untersucht, bei denen die Sammellinse zwei gleiche Krümmungsradien besizt, die Zerstreuungslinse wegen der Verkittbarkeit ebenfalls auf der einen Seite den gleichen Radius, während die andere Seite plan ist. Der Korrektionszustand ist auf der Achse ebenso gut wie bei den bestkorrigierten Dubletts, wenn man die Brechwerte für Kron und Flint im Verhältnis* 1 : 1,066 *wählt. Das System besitzt eine kleine Koma. Der Einfluss der Dicken wird diskutiert.*

1. **Introduction**. — As is well known the one radius doublet is a thin lens consisting of a cemented pair of crown and flint lenses with tolerable correction for a small field of view when used to form an image of objects at large distances. The extraordinary simplicity of its structure seems to offer advantages for its production, no spherical test glasses being needed, as the front (crown) lens has equal curvatures on both sides, so that either side will fit to the concave flint surface, while the last flint surface is a flat. It therefore seemed worth while to look more closely into the possibilities of this lens and especially to try to find out what can be expected of it and what are its limitations. The elaborate single example given by Gifford [1] in its singleness does not answer these questions. Deferring a discussion of possible practical applications to a later section, we will proceed by giving first the theoretical formulae.

2. **Thin doublet ; paraxial and third order**. — With thin systems lying in the plane of the entrance pupil we need not look into field curvatures or distortion, which leaves us only spherical aberration and coma to care for together with chromatic aberration. With the condition for a given power four variables appear to be needed. These are found in the four curvatures in the general case of an uncemented doublet. When the two components are cemented all of the conditions can be fulfilled by a proper choice of the refractive indices and dispersions [2].

The one radius doublet at first sight offers one variable, the curvature R of its spherical surfaces. Its value must be adjusted to give the required power of the system. In order to simplify the formulae we put the power of all the lenses considered here equal to unity.

The only variables available are now the refractive indices of the component glasses. Happily chromatic correction is assured by the dispersions.

We thus have the conditions :

$$(1) \qquad (2\,n - n' - 1)\,R = 1,$$

$$(2) \qquad n'_2 - n'_1 = 2(n_2 - n_1),$$

where n and n' are the refractive indices of the crown and the flint lens for the mean wave length (usually that of the D - line), and $n'_2 - n'_1$ and $n_2 - n_1$ are the differences of refractive index for the flint and crown for two determined wave lengths (usually the F- and C- lines).

To these equations we might now try to add a third one giving such a connection between n and n' that either spherical aberration or coma is corrected. The latter condition however appears not to be attainable with $n' > n$ and so we have as a last possibility the correction of third order spherical aberration, hoping

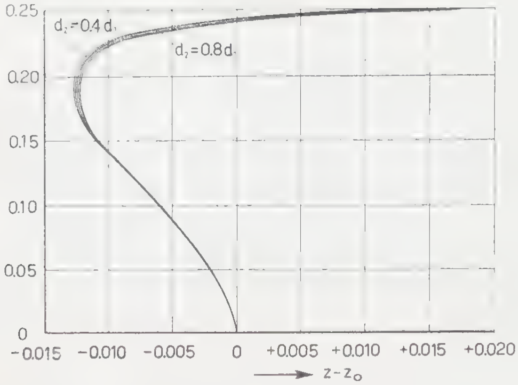

Fig. 1. — Spherical aberration variation of one radius doublet with flint thickness d_2 ; relative aperture 1/2 ; $n = 1.50$; $\alpha = 1.066$.

for the best with reference to coma correction. This leads to the following relation for n, the mean index of the crown lens, and $\alpha = n'/n$, the quotient of the mean index for flint and crown :

(3) $$\frac{8(n-1)}{n}\left\{(3n+2)(n-1)^2+n^3\right\} - \frac{n\alpha-1}{n\alpha}\left\{n\alpha+2+4(n\alpha+1)(4n-n\alpha-3)+(3n\alpha+2)(4n-n\alpha-3)^2+n^3\alpha^3\right\}=0.$$

As always happens when the refractive indices are taken as variables the degree of the equation is high. That is one of the reasons why such relations are not popular. By means of it α can be calculated for each desired value of n, and one is left with the hope that suitable pairs of glasses do occur in the glass chart.

The result is happier than might have been expected : for the range of the usual crown glasses from 1.49 to 1.54 the value of α is practically a constant ranging from 1.0661 for the low index crowns to 1.0658 for the crowns in the neighbourhood of 1.54.

We see that for one radius doublets the choice of glass is not arbitrary, and that

(4) $$n' = 1.066\, n$$

assures correction for spherical aberration.

Higher values of n are of less practical value, as the corresponding flints are so heavy that the secondary spectrum of the combinations becomes awkwardly large.

The left hand member of equation (3) multiplied by $-\dfrac{1}{32}\dfrac{b^2}{(2n-n'-1)^3}$ equals the longitudinal third order spherical aberration, where b is the diameter of the lens, a negative value of this aberration corresponding to undercorrection.

It might be remarked that the same formula (3) holds for a total power of — 1.

3. **Coma of thin one radius doublets.** — Third order coma, always present in these systems, is larger for the lower values of n. The diameter of the comatic circle in the paraxial plane has the value :

$$wb^2\left\{n^3\alpha^3-2n^2(2n-1)\alpha^2+2(2n^3-2n^2-n+1)\alpha+2n-1\right\} \Big/ 4n\alpha(-n\alpha+2n-1)^2,$$

where w is the field angle. With a tolerance of 2 % the coefficient of wb^2 can be written

$$0.41\left\{1-5.854\,(n-1.5)\right\}$$

for values of n between 1.45 and 1.55.

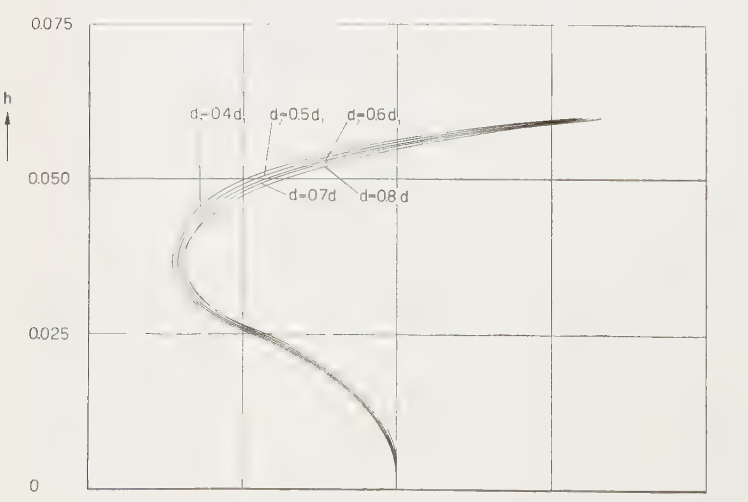

Fig. 2. — Spherical aberration variation of one radius doublet with flint thickness d_2 ; relative aperture 1/8 ; $n = 1.50$; $\alpha = 1.066$.

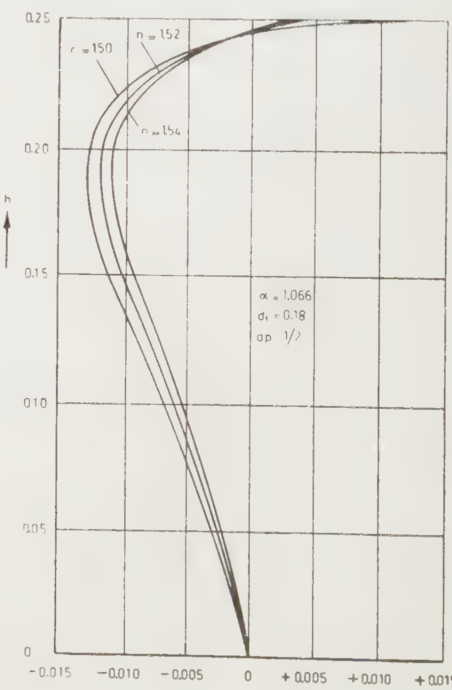

Fig. 3. — Influence of refractive index n of crown lens ; aperture 1/2 ; $\alpha = 1.066$.

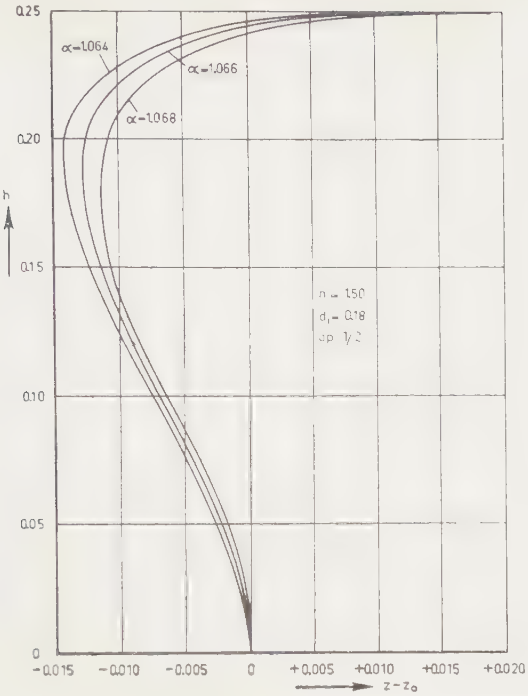

Fig. 4. — Influence of value of α; aperture 1/2; $n = 1.50$.

The value of α which annihilates the coma coefficient is in this same region of n always smaller than unity, viz. 0.961 for $n = 1.45$, 0.972 for $n = 1.50$ and 0.983 for $n = 1.55$. This excludes the simultaneous correction of spherical aberration (requiring α in the neighbourhood of 1.066) and coma.

The residual coma is small enough to make the systems useful for practical purposes. The curvature of the field and the astigmatism of the inclined pencils is large for thin systems and cannot be diminished effectively. Thus the small amount of coma is drowned in these aberrations for field angles large enough to show coma. As an example may be cited the case of a one radius doublet with relative aperture 1/10. The diameter of the comatic circle has about the same magnitude of 0.0001 of the focal length as the blurr from third order field curvature and astigmatism for a field angle of 0.02 radians.

The conclusion is that with regard to coma the one radius doublets are comparable to the usual thin object glasses.

4. **One radius doublets with thickness.** — Giving thickness to the components of the thin lens produces a change in correction. For the usual systems a more or less great alteration in the radii is used to compensate for this and to give the best form, taking into account the residual aberrations as revealed by ray tracing.

At the Delft meeting of the International Commission of Optics in 1948 T. Smith showed how a thin three radius cemented aplanatic doublet, corrected for the object at infinity, can be adjusted for an aperture b. The first curvature R_1 has to be multiplied by $1 - \frac{1}{12} b^2 R_1^2$, the second curvature has to be multiplied by $1 - \frac{1}{12} b^2 R_2^2$, and the third curvature is not changed. Up to high apertures (1 : 2) the correction appears to be the best obtainable.

In our case, however, of a one radius doublet this process does not change the form of the lens, and we can only to accept the result of the change of thickness to a appropriate value. Still it was hoped that Smith's rule would hold, i. e. that the thick doublets would give the best results, even when the form (in this case) is not changed.

Now, the ray tracings for different thicknesses of the one radius doublets show that the most optimistic expectations are fulfilled. We will give the results for the axial spherical aberration, the incidence height in a vertical direction and the corresponding distance z of the point of intersection with the axis with reference to the paraxial image point in a horizontal direction.

All the results have been reduced to focal length 1. This reduction of course is necessary, as the thickness alters the original focal length of 1 of the thin systems.

An important fact is that the spherical aberration is practically independent of the thickness of the flint lens, when this is kept within reasonable limits.

Figure 1 gives the results for an aperture of about $f/2$, figure 2 for $f/8$. In this case $n = 1.5$, $n' = 1.599$, $\alpha = n'/n = 1.066$. The crown thickness for the first case is 0.178532, for the second case 0.009643. It is evident that within a wide range of apertures the spherical correction undergoes no appreciable change

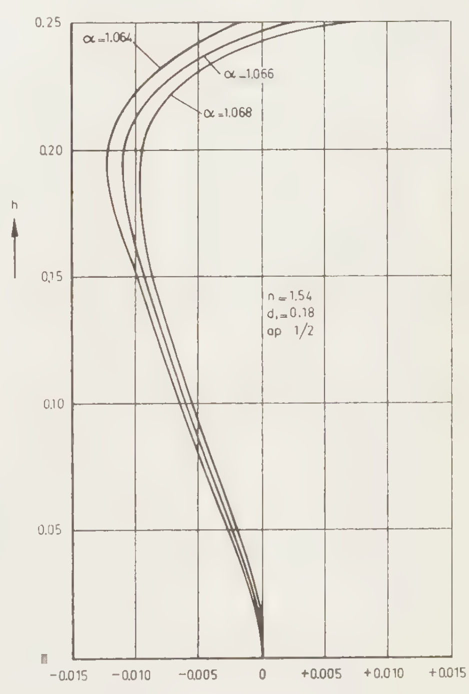

Fig. 5. — Influence of value of α; aperture 1/2; $n = 1.54$.

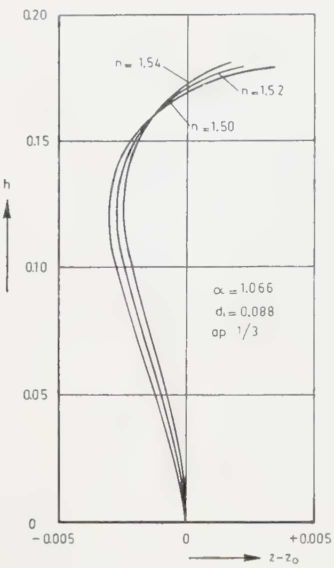

FIG. 6. — Influence of n ; aperture 1/3 ; α = 1.066.

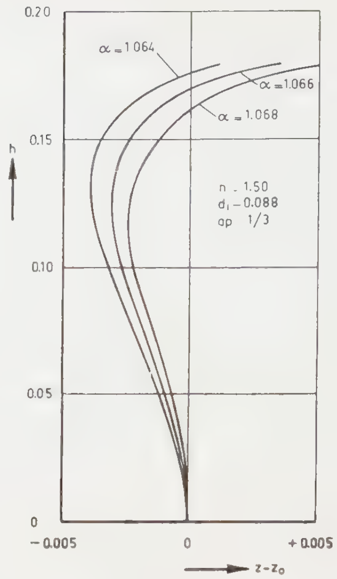

FIG. 7. — Influence of value of α ; aperture 1/3 ; n = 1.50.

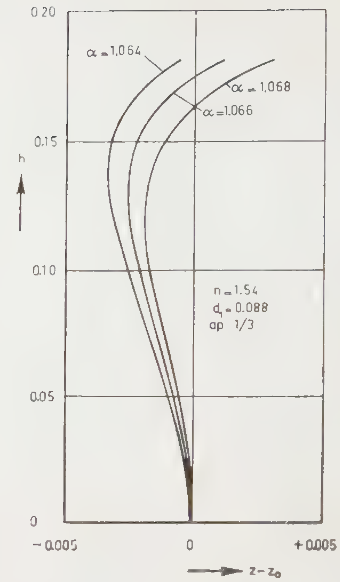

FIG. 8. — Influence of value of α ; aperture 1/3 ; n = 1.54.

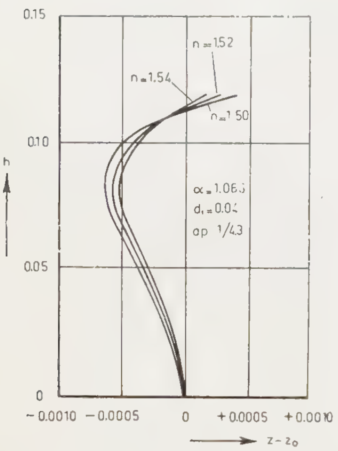

FIG. 9. — Influence of n ; aperture 1/4.3 ; α = 1.066.

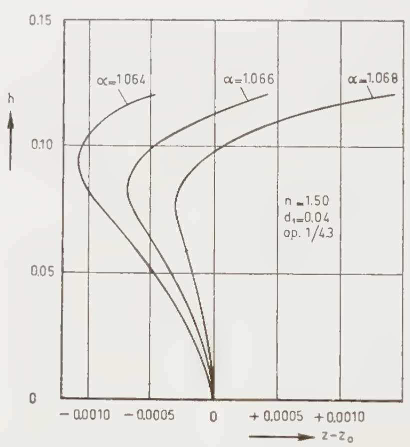

FIG. 10. — Influence of value of α ; aperture 1/4.3 ; n = 1.50.

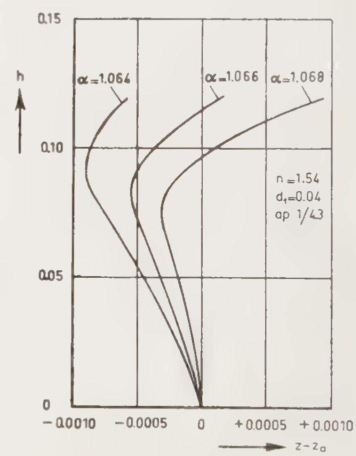

FIG. 11. — Influence of value of α ; aperture 1/4.3 ; n = 1.54.

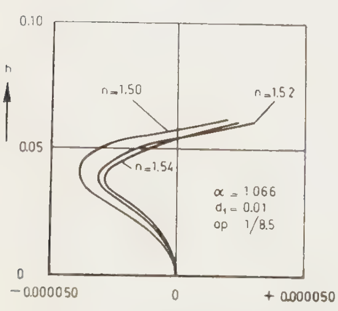

FIG. 12. — Influence of n ; aperture 1/8.5 ; α = 1.066.

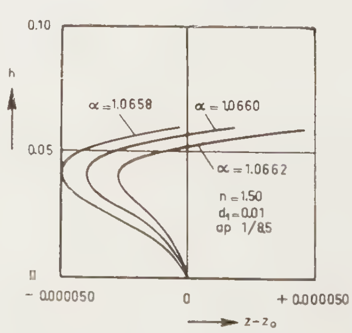

FIG. 13. — Influence of value of α ; aperture 1/8.5 ; n = 1.50.

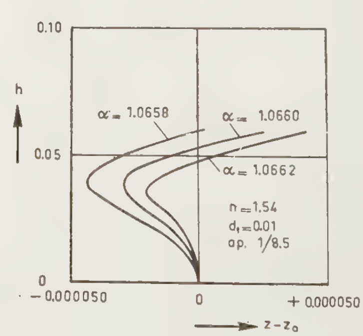

FIG. 14. — Influence of value of α ; aperture 1/8.5 ; n = 1.54.

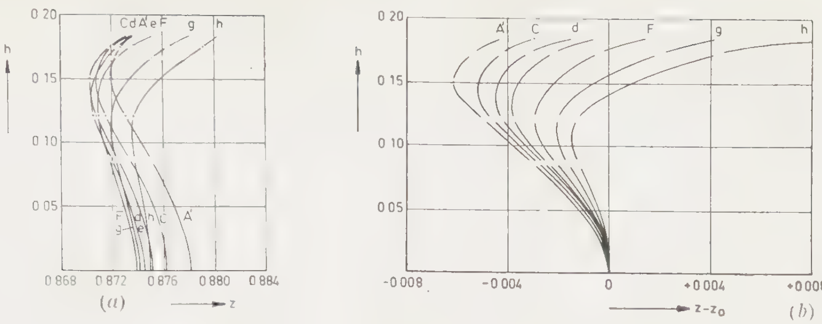

FIG. 15. — One radius doublet, 1/2.7.

with the thickness variations of the second lens. The further data therefore are given for a flint thickness of 0.6 d_1, where d_1 is the axial crown thickness.

Figures 3 to 14 contain the results of the ray tracing for different apertures (i. e. different values of d_1), and for some values of n and α. While the thickness introduces third order undercorrection, the higher orders compensate this in such a way that there is always a real corrected zone.

From these data it is possible to draw some general conclusions.

5. **General rules.** — From figures 3, 6, 9 and 12 we see that with $\alpha = 1.066$ the correction is always good for all the possible values of n. As might have been expected the residual aberrations are smaller for the higher values of n. The difference is more marked for the lower apertures.

The strictness with which α must be chosen in the immediate neighbourhood of 1.066 is strongly dependent on the aperture, as can be concluded from comparison of figures 4, 7, 10 and 13 for $n = 1.5$, and of figures 5, 8, 11 and 14 for $n = 1.54$.

For an aperture of about $f/2$ the value of α for the three curves of figures 4 and 5 differs by 0.002. The same applies to figures 7 and 8, and to 10 and 11, the change in the curve rapidly increasing with decrease of the aperture. For the smallest aperture studied, that is about $f/8.5$ the variation of α is chosen ten times smaller ; see figure 13 and 14. With the smaller variation of 0.0002 the general trend appears to be the same as with the stronger variation at higher apertures.

This means that for the best correction the tolerance for α, being rather wide for $f/2$, becomes more and more strict for the lower apertures. Of course the residual aberrations with faulty values of α are not great at these low apertures, but one appears to be wandering farther from the best conditions with a wrong value of α the smaller the aperture is.

This fact has an immediate consequence on the chromatic variation of spherical aberration. For the high apertures it is drowned in the residual spherical aberration present. At low apertures the spherical correction tends rapidly to over-correction in the violet and under-correction in the red, when α is exactly 1.066 for a wave length in the yellow or green.

The regular trend of the curves leads to the formulation of *general rules*. These are given for an equivalent focal length of $+ 1$.

The most important values can be expressed in terms of the aperture. We will take as a measure the aperture number q, meaning the number by which f must be divided in order to get the diameter of the lens. There is a choice to be made : by the diameter might be meant the largest possible diameter, which, however, is unpractical, as the rim thickness of the crown lens then is zero. Or we might take the diameter corresponding to the corrected zone, which again is unpractical as a certain value of over-correction can be tolerated for the outer zones. Or, and this is what we will do, we will define q by means of a lens diameter which is 5 % larger than that corresponding to the corrected zone.

I. The axial thickness of the crown lens is $0.71/q^2$ to within 10 % of its value.

II. A good approximation for the radius is

$$[(2 - \alpha)\, n - 1](1 + 0.137/q^2).$$

III. The largest value of zonal (axial) spherical aberration is $-0.185/q^4$ to within 10 % of its value. With $n = 1.5$ a better approximation is $-0.2\, q^4$, with $n = 1.54$ a better approximation is $-0.17/q^4$.

IV. The incidence height corresponding to the largest zonal spherical aberration is 0.75 of that corresponding to the corrected zone (to within 2 % of its value); this means that the incidence height corresponding to the largest zonal spherical aberration is 0.71

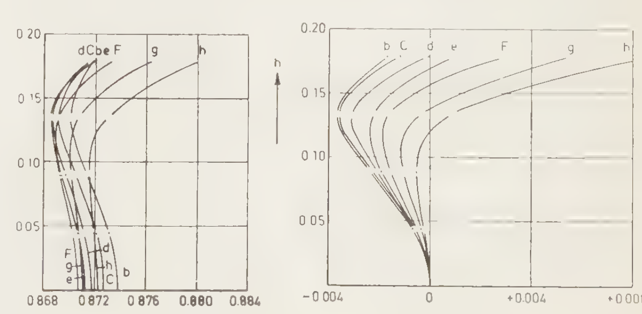

FIG. 16. — Three radius doublet, 1/2.7.

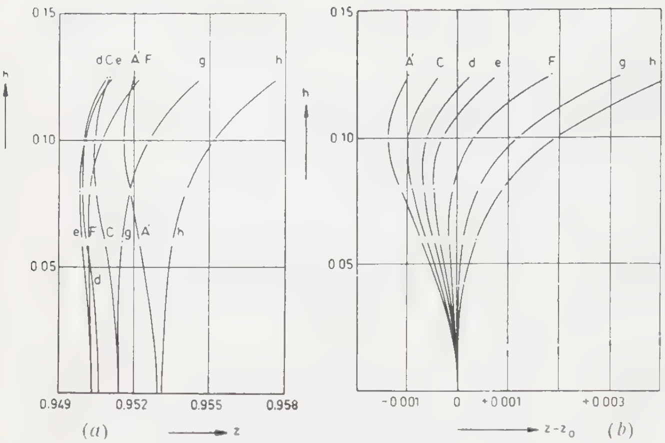

Fig. 17. — One radius doublet, 1/4.

of that corresponding to the rim distance as defined above (to within 2 % of its value).

V. The change of the axial spherical aberration for the rim zone for a variation $\Delta\alpha$ of α is $7.5\ \Delta\alpha/q^2$ to within 10 % of its value. For $n = 1.5$ a better coefficient is 8.0, for $n = 1.54$ a better coefficient is 7.0.

VI. The tolerance in the value of α in order to keep the residual aberrations within certain limits can be assumed to be $0.0061/q^2$, which keeps the change in the extreme zonal aberration less than one fourth of the rim aberration.

In order to assure the fact that the corrected zone falls within the rim as defined above, it is advisable to choose α not smaller than 1.066, except for the highest apertures.

From the above it follows that α should be within the following limits :

q	
2	1.065 to 1.0675
3	1.066 to 1.0667
4	1.066 to 1.0664
8	1.066 to 1.0661
16	1.066 to 1,066024

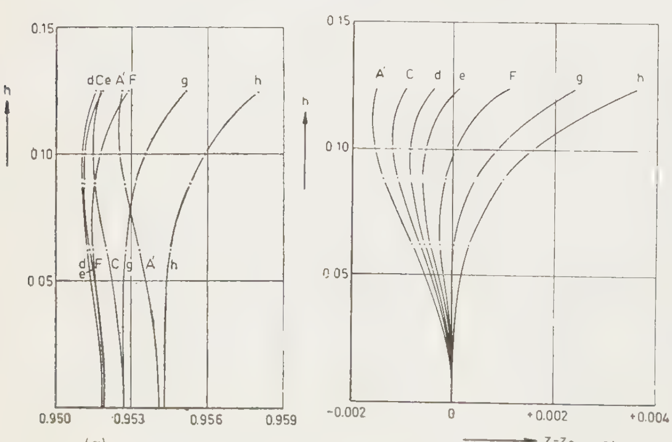

Fig. 18. — Two radius doublet, 1/4.

6. **Chromatic variation of spherical aberration ; comparison with other systems.** — In order to enable the reader to compare the behaviour of the one radius doublets on the axis with other systems, which have more degrees of freedom, we will give the results of ray tracings for different apertures in the whole range of the visible spectrum. Again the equivalent focal lengths for the d-line are + 1 throughout.

Figure 15 *a* gives the axial spherical aberration for a one radius doublet with aperture 1/2.7 and α for the d-line equal to 1.06598. The data are :

R	d	n_d	
+ 2.313 688			
	0.113 600	1.520 15	63.59
− 2.313 688			
	0.068 160	1.620 45	37.97
0			

The ratio of $n_F - n_C$ for flint and crown is 1.9976.

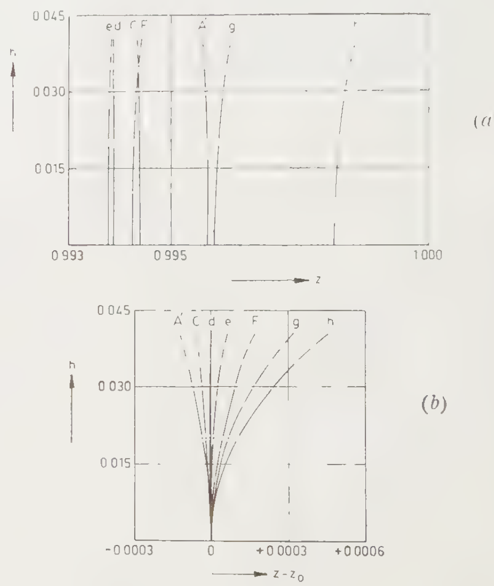

Fig. 19. — One radius doublet, 1/12.

In figure 15 *b* we give the axial spherical aberration for the different wave lengths with the paraxial focus for each wave length as zero point.

In figure 16 *a* and 16 *b* the corresponding results are given for a system with the following data [3] :

R	d	n_d	
+ 1.496 842			
	0.094 463	1.516 780	60.73
— 2.214 057			
	0.033 340	1.701 000	30.20
— 0.928 620			

Figures 17 *a* and 17 *b* relate to a one radius doublet with aperture 1/4 made from the same glasses as those of figure 15. The curvature is + 2.354 724, the thicknesses are 0.044 375 and 0.026 625.

For comparison we give in figures 18 *a* and 18 *b* the

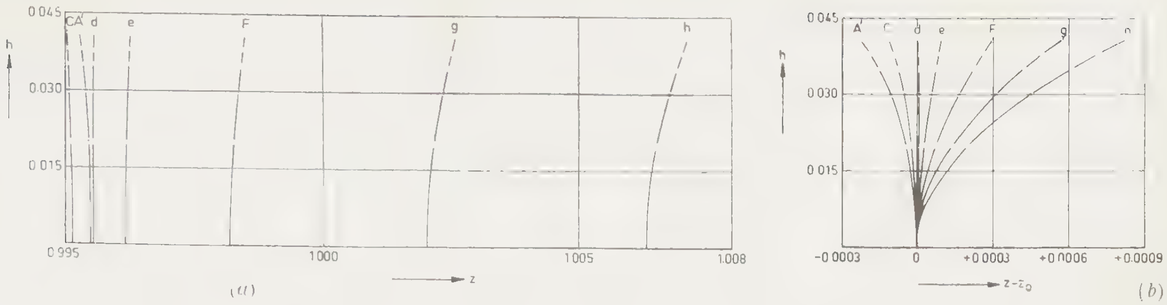

FIG. 20. - Fraunhofer object glass, 1/12.

results for the doublet mentioned by CONRADY [4], with slightly different glasses and reduced to an equivalent focal length of + 1.

R	d	n_d	
+ 2.145 204			
	0.039 393	1.515 140	63.05
— 2.145 204			
	0.026 262	1.622 500	36.19
— 0.126 925			

Lastly figures 19 *a* and 19 *b* relate to a one radius doublet with aperture 1/12 with the same glasses as those of figures 15 and 17. The curvature is + 2.378 763 the thicknesses are 0.004 930 and 0.003 944.

This system is to be compared with the celebrated FRAUNHOFER object glass [5], with the glasses BK 6 and F 4 of the SCHOTT list, and reduced to an equivalent focal length of + 1, for which the results are given in figures 20 *a* and 20 *b*.

R	d	n_d	
+ 1.458 916 9			
	0.005 060 0	1.531 130	62.05
— 3.662 467 0			
	0	1	
— 3.593 244 7			
	0.003 980 0	1.616 590	36.61
— 0.806 529 7			

It is clear that in all the cases the one radius doublet, provided the right glasses are chosen for its construction, compares favourably with corresponding other systems. It seemed therefore worth while to call attention to these simple systems, that have decided advantages in their construction, so far as relatively thin lenses are considered.

We have used the one radius doublets in several cases, for instance when an inverted image without magnification is desired. Two one radius doublets with their convex surfaces turned to one another give a satisfactory and easily manufactured solution, with the additional advantage that, on account of the symmetry of system and pencils, the residual coma is completely eliminated. Of course the drawbacks of such systems with relation to field curvature set the limit to their use when an appreciable field angle is to be covered.

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Anamorphic Mirror Systems

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Summary. — *New afocal anamorphic systems with cylindrical mirrors are described. Image quality and distortion are discussed. Means of compensating the upward horizon distortion resulting from downward projection on a curved screen are given.*

Sommaire. — *Description de nouveaux systèmes anamorphoseurs à miroirs cylindriques. Discussion sur la qualité des images et la distorsion. Etude des moyens de compenser la courbure de l'horizon vers le haut par une projection vers le bas sur un écran incurvé.*

Zusammenfassung. — *Es wird ein neues afokales anamorphotisches System aus Zylinderspiegeln beschrieben. Bildgüte und Verzeichnung werden diskutiert. Man kann dabei auch die Verzerrungen kompensieren, die sich aus der Projektion auf einen zur optischen Achse geneigten Bildschirm ergeben.*

1. **Introduction.** — The use of anamorphic systems in optics is very old. As far back as the first half of the last century Brewster [1] developed a set of prisms with anamorphic action. With two of such sets turned at an angle of 90° he made an enlarging telescope of small power.

A Zeiss patent of Rudolph of 1899 [2] describes anamorphic systems composed of cylindrical lenses with crossed generatrices, whereas Burch [3] in 1904 gives an afocal combination of cylindrical lenses with parallel generatrices. The latter system is on the analogy of an astronomical telescope if the " eye piece " is positive, and of a Galilean telescope if it is negative.

Chrétien [4] in 1930 applied anamorphic systems to cinematography. He improved the lens system of Burch which he placed in front of the objective of a camera, thus producing a «squeezed» image which was drawn out to normal proportions by applying a similar lens system between the projection lens and the screen.

This paper deals with a third group of anamorphic systems in which cylindrical mirrors are employed [5]. The use of mirrors has two great advantages, the most obvious of which is the absence of chromatic aberrations. Moreover, the spherical aberration of a cylindrical mirror is very small as compared with that of a cylindrical lens with the same aperture and the same focal length. It proved possible to obtain very good image definition with a system composed of two mirrors only. These systems are designed, patented and manufactured by the *Old Delft* at Delft, Holland, and are called *Delrama Systems.*

2. **The First Delrama System.** — The most obvious method of making an anamorphic system of zero power by means of two cylindrical mirrors is indicated in figure 1, showing a section perpendicular to the generatrices of the cylinders. M_1 is a concave, and M_2 a convex mirror of which the focal points F coincide on the optical axis FA. The anamorphic factor β is, as in the case of the Chrétien lens, the ratio of the focal lenghts of the two components.

The mirror M_1 produces under- and M_2 overcorrection, the aberration of the former being predominant.

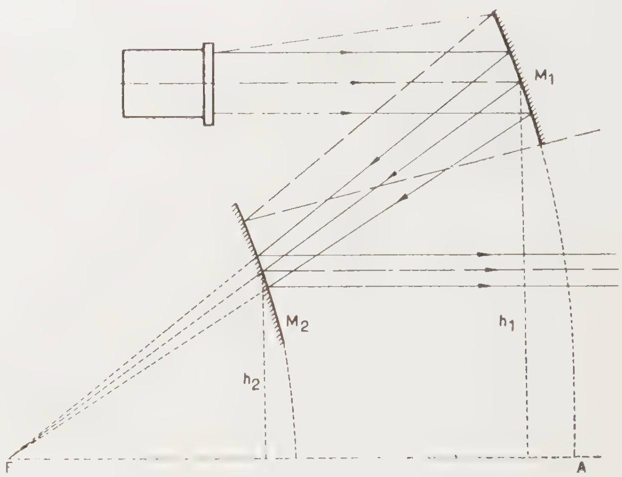

Fig. 1. — Delrama I.

The « eccentric » distance $h_1 - h_2$ must be so great that there will be no vignetting in the field (see dash-dotted lines).

In consequence of this great distance the amount of aberration is fairly large, but can be reduced by shifting the two mirrors with respect to each other in such a way that two points of the caustical curves of M_1 and M_2 coincide.

Reasonably satisfactory definition may be obtained in this way for limited fields and anamorphic ratio ; but a perfect definition can be achieved by adding a cylindrical lens, one surface of which is either the concave or the convex mirror. Such a solution is indicated in figure 2.

The focussing at finite distances is achieved by shifting one of the mirrors along the optical « axis ».

3. **The Second Delrama System.** — An apparently much less obvious solution with reduced aberrations is indicated in figure 3. Here two sections are drawn, I being parallel to, and II perpendicular to the generatrices of the two cylindrical mirrors M_1 and M_2. In the last section it can be seen that the incident height

h of the border ray is reduced by a factor of about 6 as compared with figure 1. As the spherical aberration is proportional to the third power of h, the reduction is by a factor of about 200. The resulting aberrations for beams parallel to the optical axis and for-

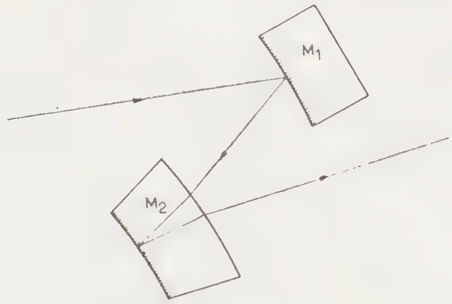

Fig. 2. — Delrama Ia.

ming angles of 3°54′ and 7°45′, corresponding to the half field and whole field of an unconventional film (CinemaScope), in combination with a projection lens of 85 mm focal length are shown in figure 4. The radii of curvature are 538 mm for the concave and 269 mm for the convex mirror, giving an anamorphic ratio of $\beta = 2$.

For beams not perpendicular to the generatrices of the cylinders the aberrations remain the same. This is a consequence of a property of mirror reflection, as the geometrical projection of the incident and the reflected rays on the plane II (fig. 3) is independent of the angle between the rays and this plane (1).

(1) Here and often in the following paragraphs the word " projection " is used in a geometrical sense, and is to be distinguished from the light projection of the image on a screen.

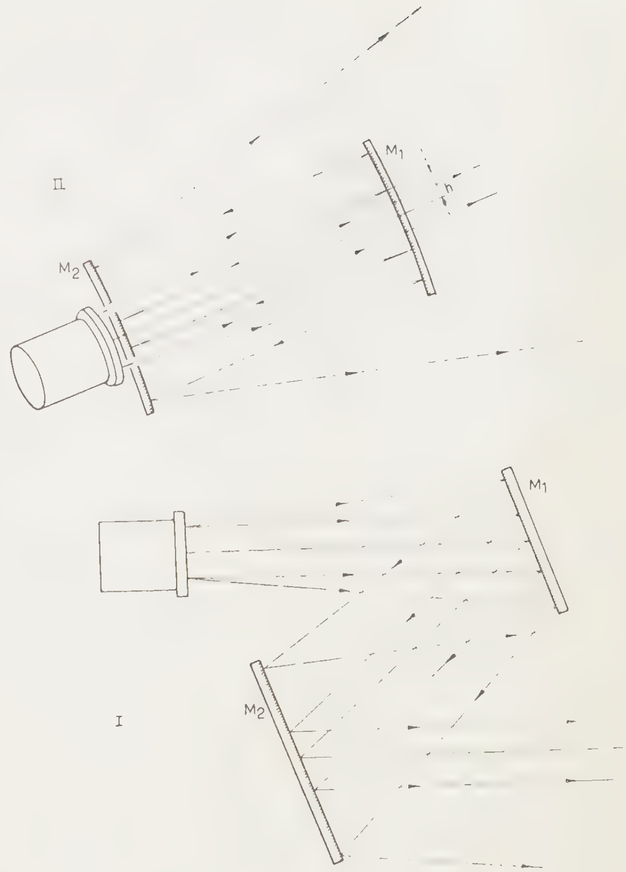

Fig. 3. — Delrama II.

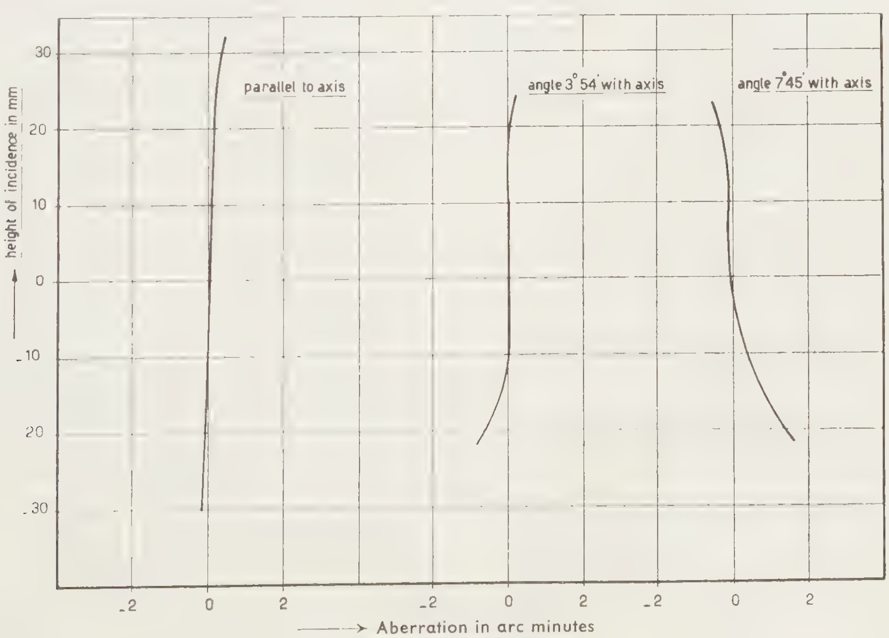

Fig. 4. — Aberration of the Delrama II.

4. **Distortion.** — The anamorphic system of figure 3 produces a distortion of the image which is in many cases advantageous to compensate the distortion resulting from downward projection on a curved screen in cinema theatres. We will revert to this question later on.

This section is intended to discuss the degree of distortion and how it depends on the factors involved.

Our starting point is the fact already mentioned in the preceding paragraph that geometrical projection of light rays on a plane perpendicular to the generatrices is independent of the angle between these rays and the plane of projection.

In figure 5 the two mirrors M_1 and M_2 have the common focus F. The screen S is considered to be at a large distance a_0 from F, parallel to the vertical generatrices of M_1 and M_2.

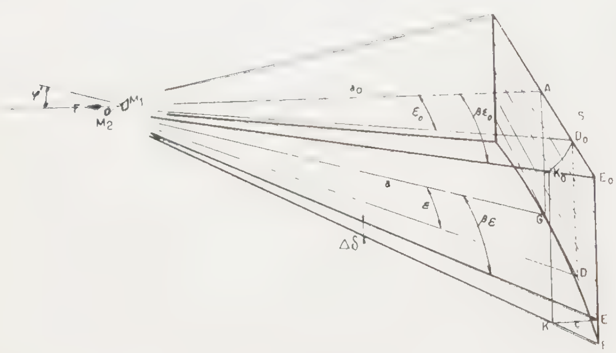

FIG. 5. — Distortion.

The principal ray FD_0 of a substantially horizontal beam would intersect the screen S at a point D_0 if the mirrors M_1 and M_2 were absent. Because of the presence of the anamorphic system this ray is diverted to FE_0 intersecting the screen S at E_0 in such a way that $AE_0 = \beta\, AD_0$, where β is the anamorphic magnifying ratio.

Now we will consider another incident ray FD having the same projection FD_0 on the horizontal plane FAD_0 and making an angle φ' with this plane. Its point of intersection with the screen is E, and

$$(1) \quad p_0 = E_0E - AG = (FE_0 - FD_0)\, \mathrm{tg}\, \varphi' \quad .$$

As now in figure 5

$$FD_0 = a_0/\cos \varepsilon_0 \text{ and } FE_0 = a_0/\cos \beta\varepsilon_0$$

we find

$$(2) \quad p_0 = a_0\, \mathrm{tg}\, \varphi' \left(\frac{1}{\cos \beta\varepsilon_0} - \frac{1}{\cos \varepsilon_0}\right) \approx \frac{a_0}{2}\, \varepsilon_0^2 (\beta^2 - 1)\, \mathrm{tg}\, \varphi' .$$

Here ε_0 is the projection on the plane FAD_0 of the field angle ε. Now if the incident ray lies in a plane forming an angle φ with the plane FAD_0, then

$$\mathrm{tg}\, \varphi' = \mathrm{tg}\, \varphi \cos \varepsilon_0 \approx \mathrm{tg}\, \varphi$$

for small values of ε_0.

All incident principal rays lying in this plane will intersect the screen in a straight line GD, the deviated rays, however, in a curved line GE, which is to a first approximation a circle with radius r_0.

The reciprocal of r_0 is given by

$$\frac{1}{r_0} = \frac{2\, p_0}{(GE')^2}, \text{ and as } GE' = a_0\, \mathrm{tg}\, \beta\varepsilon_0 \approx a_0\, \beta\varepsilon_0$$

we find

$$(3) \quad \frac{a_0}{r_0} = \left(1 - \frac{1}{\beta^2}\right) \mathrm{tg}\, \varphi \quad .$$

The distortion described by relation (3) may also be understood as a consequence of image curvature. In reality we are concerned with a beam of incident rays converging to the screen at the point D situated a distance $a_0\, \mathrm{tg}\, \varphi$ below the point D_0 of figure 5. In a plane parallel to the generatrices of the cylinders M_1 and M_2 the convergence of the rays is not influenced. Thus the deviated rays will converge to K after having covered in a horizontal direction the distance $FK_0 = FD_0$. For small values of the angle ε the points K and K_0 are situated in front of the screen at a distance c

$$(4) \quad c = a_0 \left(\frac{1}{\cos \beta\varepsilon_0} - \frac{1}{\cos \varepsilon_0}\right) = \frac{a_0}{2} \left(1 - \frac{1}{\beta^2}\right)$$

corresponding to a " radius of curvature " r_{00} of the image given by

$$(5) \quad \frac{a_0}{r_{00}} = 1 - \frac{1}{\beta^2} \quad .$$

Formula (3) gives the distortion radius r_0 if the screen S is parallel to the generatrices of the anamorphic system. In general, however, this is not the case. If the screen is perpendicular to the central principal ray FG (and thus forms an angle φ with the generatrices of M_1 and M_2) the distortion curvature must be multiplied by a factor $\cos \varphi$, and if the screen is now tilted through an angle γ it must again be divided by $\cos \gamma$ (fig. 8). If the distance FG is indicated by a, then, because of $a = a_0/\cos \varphi$, we obtain instead of the relation (3) the new formula for the radius r on the tilted screen

$$(6) \quad \frac{a}{r} = \left(1 - \frac{1}{\beta^2}\right) \frac{\mathrm{tg}\, \varphi}{\cos \gamma} \quad .$$

In cinema projection the distortion will be different for horizontal lines at different heights. If half the projection angle in a vertical direction is denoted by ψ, we find by applying (2) and (6) for the relative curvature a/r_t of the top horizontal lines and a/r_b of the bottom horizontal lines

$$(7) \quad \begin{aligned} \frac{a}{r_t} &= \left(1 - \frac{1}{\beta^2}\right) \frac{\mathrm{tg}\, (\varphi - \psi)}{\cos \gamma} \\ \frac{a}{r_b} &= \left(1 - \frac{1}{\beta^2}\right) \frac{\mathrm{tg}\, (\varphi + \psi)}{\cos \gamma} \quad . \end{aligned}$$

5. **All-glass Delrama System**. — Instead of taking two cylindrical mirrors M_1 and M_2 one may employ a solid glass body in the shape of a rhombohedron of which the two oblique surfaces M_1 and M_2 are shaped cylindrically with the generatrices parallel to the plane of the drawing (fig. 6).

Here again the anamorphic ratio β equals the ratio of the radii of curvature.

Focussing at finite distances is achieved by dividing the rhombohedron by a nearly horizontal plane into two prisms and by shifting one of the prisms along the dividing surface, thus diminishing the distance between the reflecting surfaces M_1 and M_2.

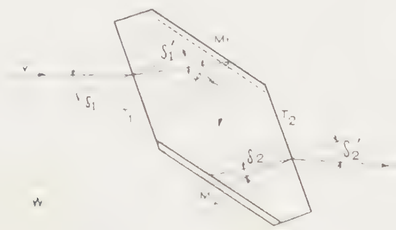

FIG. 6. — All-glass Delrama II.

This all-glass system has several properties differing from that of figure 3, all of which have the same cause, viz., the refraction of skew rays on the plane entrance and exit surfaces of the rhombohedron.

As beams corresponding to the border of the image form in the glass a much smaller angle with the central ray than in air, the dimensions may be reduced. Furthermore, the distortion is different. To show this we will discuss the distortion of rays incident in a plane V which is perpendicular to the plane W of figure 6.

V forms an angle δ_1 with the normal to T_1. The direction of an incident ray in V is determined by the angle ε_1 with the plane W of the drawing. The angles which the projection of the ray on the plane W forms after refraction at T_1 and T_2 and reflection at the cylindrical mirrors M_1 and M_2 are indicated in figure 6. We now wish to know the deviation

$$\Delta\delta_2' = \delta_2' - \delta_{02}'$$

in which δ_2' and δ_{02}' are the projections on the plane W of the exit angles of two rays the first of which forms before the entrance at T_1 an angle ε_1 with the plane W, while the second is situated in W. The deviation $\Delta\delta_2'$ is the sum of three contributions, viz. two on account of the refractions at the parallel planes T_1 and T_2, and the third because of the reflection at the pair of anamorphic mirrors M_1 M_2. The contribution to the deviation at T_1 is a consequence of the laws of refraction, according to which (*)

(*) see CZAPSKI-EPPENSTEIN : Grundzüge der Theorie der optischen Instrumente, 3e Auflage 1924, page 6 and 330-334.

$$\text{(8)} \qquad \cos\varepsilon_1 \sin\delta_1 = n \cos\varepsilon_1' \sin\delta_1'$$

$$\text{(9)} \qquad \sin\varepsilon_1 = n \sin\varepsilon_1' \quad ,$$

where n is the index of refraction of the glass, while ε_1 and ε_1' are the angles of the incident and refracted rays with W.

If δ_{01}' is the value of V_1' for a principal ray situated in the plane W, we find by expanding (8) in a series of ε_1 and ε_1', taking into account only the quadratic terms, and applying (9)

$$\text{(10)} \quad \Delta\delta_1' = \delta_1' - \delta_{01}' = -\frac{\varepsilon_1^2}{2}\left(\frac{n^2-1}{n^2}\right) \operatorname{tg}\delta_1' \quad .$$

After reflection at M_1 and M_2, there is added to this deviation the angle $\Delta\delta$ given by

$$\text{(11)} \qquad \Delta\delta = \frac{\varepsilon_1'^2}{2}(\beta^2-1)\operatorname{tg}\varphi \quad .$$

This relation is equivalent to (2) in a medium with index of refraction n as in figure 5 the distortion angle $\Delta\delta$ is for small angles ε equal to

$$\Delta\delta = \frac{p_0 \cos^2\varphi}{a_0} \text{ and } \varepsilon = \varepsilon_0 \cos\varphi' \quad .$$

Finally, on account of the refraction at T_2 there is added a deviation with opposite sign from that of the surface T_1 and which is β^2 times as great, as the angle between the ray and W is now enlarged by a factor β due to the anamorphic system of mirrors M_1 and M_2.

The combination of the three deviations gives, after a simple computation

$$\text{(12)} \quad \Delta\delta_2' = \frac{\varepsilon_1^2}{2n}(\beta^2-1)\frac{\cos\delta_{02}}{\cos\delta_{02}'}\left\{\operatorname{tg}\varphi + (n^2-1)\operatorname{tg}\delta_{02}\right\}$$

from which follows, for the relative curvature of a horizon projected on a screen perpendicular to the planes V and W, the expression

$$\text{(13)} \quad \frac{a}{r} = \left(1-\frac{1}{\beta^2}\right)\frac{1}{n}\frac{\cos\delta_{02}}{\cos\delta_{02}'}\left\{\operatorname{tg}\varphi + (n^2-1)\operatorname{tg}\delta_{02}\right\} .$$

From these formulae we may conclude :

1) In the case of $\delta_1 = 0$ and hence the plane V of the incident rays perpendicular to the refracting surfaces T_1 and T_2, the distortion is reduced by a factor equal to the index of refraction n as compared with the mirrors in air [see relation (6)].

2) It is fundamentally possible to compensate the distortion of the pair of mirrors M_1 M_2. We will use this property to obtain a distortion-free Delrama, to be described hereafter.

3) In the limit when the planes T_1 and T_2 are parallel to the generatrices of M_1 and M_2 ($\varphi = \delta_{02}$) we obtain an expression equivalent to (6) which is independent of the index of refraction.

At first sight one would think that the incorporation of glass would introduce a noticeable chromatic aberration as a consequence of the different entrance and exit angles of the rays. This, however, is not the case. One can see this approximately as follows : if

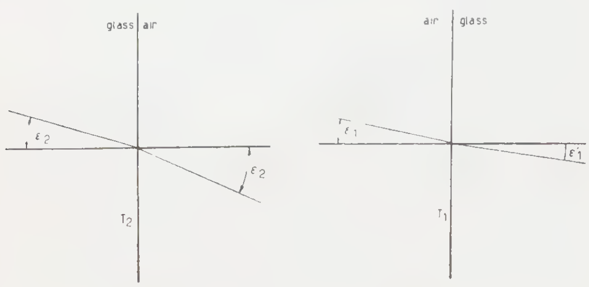

FIG. 7. — Chromatic aberration.

the plane V is perpendicular to T_1 and the incident rays impinge on the glass at an angle ε_1, the refracted rays form an angle ε_1' given by (see fig. 7)

$$\sin \varepsilon_1 = n \sin \varepsilon_1' \quad . \tag{14}$$

This angle ε_1' will be increased to ε_2 by reflection at the surfaces M_1 and M_2, so that

$$\sin \varepsilon_2 = \beta \sin \varepsilon_1' \tag{15}$$

and after refraction at T_2

$$\sin \varepsilon_2' = n \sin \varepsilon_2 \quad . \tag{16}$$

Thus we find

$$\sin \varepsilon_2' = \beta \sin \varepsilon_1 \tag{17}$$

and since the index of refraction does not enter into the relation (17), there will be no colour shift in a horizontal direction. However, the relation (15) is not exactly valid and thus there will be a small chromatic aberration. Exact calculation shows that this aberration amounts to less than 10 seconds of arc between red C and blue F lines for the extreme rays in the case of a rhombohedron system as incorporated in a Delrama with an anamorphic factor of 2 used in combination with a lens $f = 50$ mm.

The colour shift in a vertical direction follows from relation (12) by differentiation with respect to the index of refraction n.

For the same extreme rays as considered above we find a difference of 0.5 minute of arc, corresponding to only 7μ on the film, which is confirmed by exact calculation. The chromatic aberrations are thus negligibly small.

6. **Distortion in the case of oblique projection on a curved screen and its compensation.** — In many theatres the projection is oblique downwards. As a result the image of the horizon when projected on a curved screen is distorted upwards. This distortion can be compensated by applying a Delrama System with a downward distortion. The condition for this compensation for a spectator in the theatre will now be deduced.

In figure 8 the projector P causes an image on the screen S, of which B is the centre. The screen is cylindrically curved with a radius of curvature equal to ρ_0 and is tilted, the normal forming an angle γ with the line PB. The spectator is at A, the angle ABP being α.

If the spectator does not see the horizontal edges of the screen a projected horizon at B will give an impression of a straight line if all the lines connecting A with this horizon are in a flat plane U. In other words : compensation of distortion is obtained when the flat plane U intersects the cylindrical screen S in the same ellipse as the projected horizon. The latter is the intersection of S with a cone having its apex at P ; this must be identical with the intersection of the cone with U.

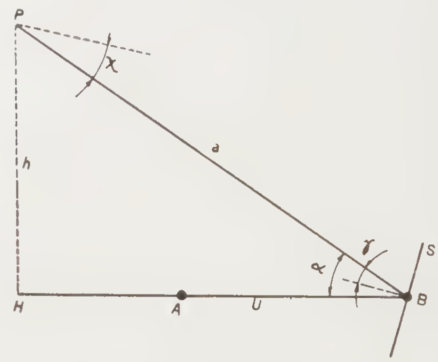

FIG. 8. — Distortion in the case of oblique projection on a curved screen.

The radius of curvature ρ of the circle osculating the intersection ellipse of U and S at its apex is given by

$$\rho = \rho_0 \cos(\alpha - \gamma) \tag{18}$$

The radius r_u of the circle osculating the intersection of U with the projected cone is given by

$$r_u = r \sin \alpha \tag{19}$$

if we define r as the radius of curvature of the horizon, projected on a flat screen perpendicular to the line PB. The expression for the relative distortion a/r is given in (13), in which a is the projection distance.

Now compensation is attained if $r_u = \rho$, thus giving the relation

$$\rho_0 \cos(\alpha - \gamma) = r \sin \alpha \quad . \tag{20}$$

From (13) and (20) the values of the angles φ and δ_{02} can be calculated at given values of the anamorphic magnification β, the angles α and γ and the radius ρ_0.

In many practical cases $\cos(\alpha - \gamma)$ is to a sufficient approximation equal to 1. Moreover, the radius of curvature ρ_0 of the screen S is often assigned a value equal to the projection distance a. Then (20) changes to

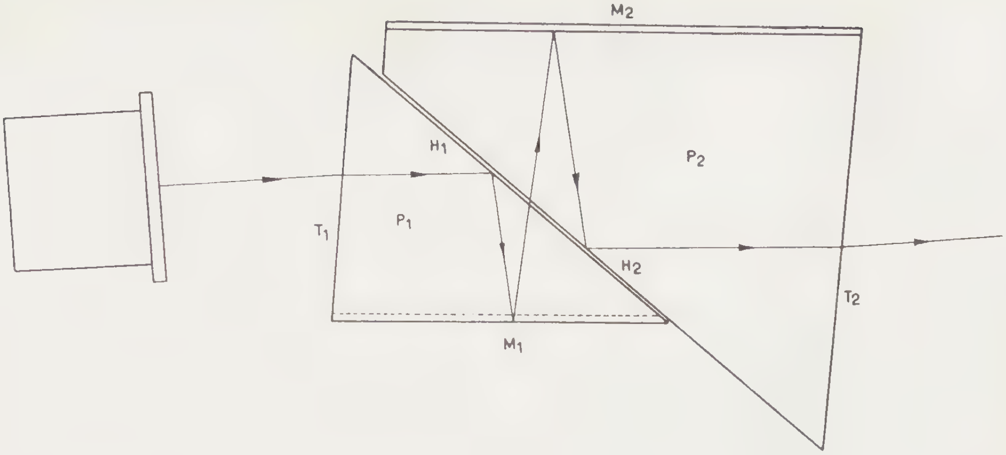

FIG. 9. — Delrama IV.

(21) $$\sin \alpha = \frac{a}{r} \quad .$$

Let us take as an example a theatre with the following characteristics.

Distance $a = 20$ metres.

Downward projection angle $\chi = 14°$.

Focal length of objective = 100 mm.

Seat of a spectator A at the centre and at the same horizontal level as the bottom part of the screen.

As the distance from A to S is 10 m, we find for the angle α a value of 24°. In the case of the two-mirror Delrama as sketched in figure 2, having an anamorphic ratio $\beta = 2$, we find for the required angle φ :

$$\operatorname{tg} \varphi = \frac{3}{4} \sin 24°, \text{ and hence } \varphi = 28° \quad .$$

If the compensation is realized for the centre B of the screen, the projection of the upper horizontal line will be seen curved upwards and that of the lowest line downwards. Now in most theatres the faintly illuminated edges of the screen can be observed. The psychological impression of the curvature of horizontal lines is much influenced by whether these are parallel or not to the edges of the screen, parallelism helping to give an impression of the absence of distortion. Thus best results will probably be obtained by making the horizontal edges of the screen coincide with the top and bottom projected horizontal lines of a film taken with an anamorphic system in combination with an objective of 50 mm (*). In this way the inherent distortion of the horizontal extreme lines in the film image is also dealt with.

7. **Distortion-free Delrama System.** — The distortion which is an advantage when projecting downwards in a cinema theatre is a disadvantage in other circumstances and especially during the taking of a film. It can, however, be eliminated in a simple way.

(*) The distortion of the film image is dependent on the field angle and hence on the focal distance. Lenses with 50 mm focal length are used in most exposures.

Moreover, it is then possible to deal with the shifting of the rays, which amounts to 12 cm in the case of the two-mirror projection Delrama for 35 mm film having an anamorphic ratio of 2.

The optical system may be described with the aid of figure 9. The light from the projector enters a prism P_1 at the plane surface T_1, is totally reflected at the hypotenuse H_1, and afterwards at the concave cylindrical mirror M_1, can now pass the hypotenuse because of the change of angle due to the reflection at M_1 and enters prism P_2. Then the light is reflected at the convex cylindrical surface M_2 and afterwards by total reflection at the hypotenuse H_2, leaving the system at T_2.

Focusing at finite distances is obtained by shifting the prism P_1 along the surface H_2 thus altering the distance between M_1 and M_2.

According to (13) the distortion is eliminated if

(22) $$\operatorname{tg} \varphi + (n^2 - 1) \operatorname{tg} \delta_{02} = 0 \quad .$$

If a system with an anamorphic ratio β equal to 2 suitable for a lens with focal length of 50 mm and a film image of 23.2×18.2 mm² is made of borate crown glass ($n = 1{,}517$) the angle φ amounts to 8°15′ and we find from (22) for δ_{02} the value 6°15′ or for δ_{01} 9°30′ (*).

8. **Comparison of the distortion of other systems.** — In the preceding section the term " distortion-free " has been used. This term refers to the property of a system of representing a horizontal straight line at the centre of the film image after projection on a flat screen as an actual horizontal straight line. However, the projection of horizontal lines at the bottom or the top of the image will then be curved in accordance with (12) and (13). This distortion is also a property

(*) The curvature of the horizon can also be eliminated by combining two Delrama Systems with opposite distortion. If the relative distortions of the first and the second system are $(a/r)_{I}$ resp. $(a/r)_{II}$, and the anamorphic ratio of the second is β_{II}, we find :

$$a/r = \left\{ (a/r)_{I} + \beta_{II}^2 (a/r)_{II} \right\} / \beta_{II}^2$$

of the anamorphic systems composed of cylindrical lenses or of BREWSTER prisms ; the relative distortion curvature is in these cases also given by (6) if φ is the angle between the rays and a plane perpendicular to the generatrices of the cylindrical lenses or to the parallel edges of the prisms. The same derivation of (6) may be given as the angle φ will remain the same after the light ray has passed through the system.

FIG. 10. — Delrama II 35 mm.

There is, however, a difference between the Delrama System and the others with respect to the distortion of vertical lines. The Delrama of figure 2 with the two mirrors in air provides completely straight vertical lines in the image as follows immediately from the fact that the geometrical projection of the rays on a plane perpendicular to the generatrices of the mirrors is independent of the angle of the rays with these generatrices.

With the all-glass Delrama of figure 5 or 9 there is a very small distortion, which can be deduced along the same lines as the relation (13).

We find :

$$\frac{a}{r_\perp} = n^2 (n^2 - 1) \left(\frac{\cos \varepsilon_2}{\cos \varepsilon_1'} - 1 \right) \operatorname{tg} \varepsilon_2' \tag{23}$$

in which $r_\perp$ is the radius of curvature of a projected vertical line on a curved screen with radius equal to a and which is perpendicular to the central ray of the image.

For the most disadvantageous case of a Delrama for 35 mm film with anamorphic ratio 2 as given in figure 9 in combination with a lens with focal distance of 50 mm, we find for the relative curvature of a border line $a/r_\perp = 0.06$, corresponding to a linear aberration in the film image of only 0.05 mm.

With an anamorphic system consisting of two thin cylindrical lenses of refractive index n we find

$$\frac{a}{r_\perp} = \left(1 + \frac{1}{n} \right) (\beta - 1) \operatorname{tg} \varepsilon. \tag{24}$$

If $n = 1.5$ this corresponds to $a/r_\perp$ equal to 0.38 for the same image lines as above.

In principle it must be possible to correct this curvature. A cylindrical lens system as used in cinema projection proved not to be corrected in this respect, the curvature also being 0.38. This rather pronounced curvature, however, constitutes no inconvenience in most cases.

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Improved methods for producing interference filters

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SUMMARY. *The production of all-dielectric interference filters necessitates an accurate method of controlling layer thickness. A technique is described which is a modification of that due to* GIACOMO *and* JACQUINOT, *and which enables such filters to be constructed without the use of control surfaces. With this method the position of a filter pass-band may be located to within* ± 20 Å.
The useful area of a filter is limited by the non-uniform thickness of evaporated layers. Results are given of the increase in this area made possible by rotating the glass substrate during deposition.

SOMMAIRE. — *La production de filtres d'interférences, uniquement diélectriques, nécessite une méthode précise pour le contrôle de l'épaisseur des couches.*
La technique décrite est une modification de celle de GIACOMO & JACQUINOT ; *elle permet de fabriquer ces filtres sans utiliser de surfaces témoins. Avec cette méthode, la bande transmise par un filtre peut être déterminée à* ± 20 Å *près.*
La surface utile du filtre est limitée par le manque d'uniformité de l'épaisseur des couches déposées. Des résultats sont donnés, concernant l'augmentation de cette surface, rendue possible en faisant tourner le support-verre pendant l'évaporation.

ZUSAMMENFASSUNG. — *Die Herstellung von Interferenzfiltern, die ausschliesslich aus Dielektrika aufgebaut sind, verlangt eine genaue Methode in der Einhaltung der Schichtdicken. In Abänderung einer Methode von* GIACOMO *und* JACQUINOT *wird ein Weg beschrieben, der die Herstellung der Filter ohne die Benutzung von Probeflächen erlaubt. Nach dieser Methode kann die Lage des Filterbandes mit einer Genauigkeit von ± 20 Å eingehalten werden.*

Der brauchbare Bereich der Filterfläche ist bestimmt durch die Ungleichmässigkeiten in der Dicke der aufgedampften Schichten. Lässt man aber den Glasträger während des Niederschlagens rotieren, so kann man die brauchbare Filterfläche vergrössern.

In the preparation of optical filters by the vacuum evaporation of multiple dielectric films, there are two major problems — control of the thickness of the layers as they are deposited, and the production of films whose thickness is uniform over sufficiently large areas.

Control Requirements. — If it is only desired to construct multi-layer reflecting films, errors of a few per cent in the thickness of each layer can be tolerated without an appreciable loss in reflectivity [1]. If, however, all-dielectric interference filters are to be constructed to transmit a given wavelength, the thickness of the spacing layer between the reflecting films must be controlled much more accurately. At normal incidence the wavelengths of the pass-bands are connected with the thickness t and refractive index μ of the spacing layer by the relation

$$2 \mu t = n \lambda \ (n = 1, 2, 3... \text{etc.}).$$

The pass-band may be adjusted to shorter wavelengths by tilting the filter so that incidence is no longer normal, but the performance of the filter, particularly in a convergent beam, deteriorates as the tilt increases. For many applications a practical limit to the wavelength displacement obtainable by tilting is 50 Å at, say, λ 5 000.

Applying this criterion, if a batch of filters is made for which the probable error of thickness control is only 1 %, then about three-quarters of the filters will be unusable at the designed wavelength.

Existing Techniques. — Previously described methods of thickness control include visual colour matching and photoelectric measurements of reflectivity or transmission [2]. Using the former method BANNING [3] has prepared 7-layer films of ZnS and Cryolite, having a reflectivity of 94 %, which approaches the theoretical value of 96 %, but this method is insufficiently accurate for the construction of filters for a specified wavelength and in any case requires a separate control surface to measure the thickness of each layer deposited.

Most workers now appear to use photoelectric measurements. JARRETT [4] has observed the maxima and minima in reflectivity with monochromatic light from a mercury lamp and a photocell-galvanometer system, and finds that this gives errors in layer thickness of about 4 %. It is usual with this type of control to introduce a separate control surface for each layer evaporated. We have tried a method in which light from a monochromator, chopped at a suitable frequency, was passed through the glass on which the films were being deposited and the transmission measured using a photo-multiplier, amplifier and rectifier. As many as ten layers were controlled successively in this fashion without control plates, but in making interference filters we found it necessary to use a second control surface to measure the layers following the spacer. For a number of filters made in this way the probable error of reproducibility of the wavelength of the pass-band was about 2 %.

New Control System (1). — In our present system the beam of light used for control has its wavelength continually modulated over a small range. When the thickness of a reflecting or spacing layer is correct for a given wavelength, the transmission considered as a function of wavelength is stationary at that wavelength ; this gives a null indication in our instrument. GIACOMO and JACQUINOT [5] have described a method of this type which they used to measure the performance of interference filters and subsequently to produce reflecting films. They estimate that when a wedge of cryolite whose thickness varies between ·2 and ·8 μ is deposited on glass and illuminated by monochromatic light of wavelength 5 400 Å, the position of dark and bright fringes can be located to an accuracy corresponding to changes in thickness of $\lambda/300$ and $\lambda/150$ respectively. This will be the order of control accuracy obtainable for a single quarter wave layer deposited on to a control surface. They do not, however, give any indication of the behaviour of their system when successive layers are deposited on to the monitored surface, nor do they appear to have constructed all-dielectric interference filters using their apparatus.

Our control system is shown diagrammatically in

(1) Prov. Pat. No. 19109.

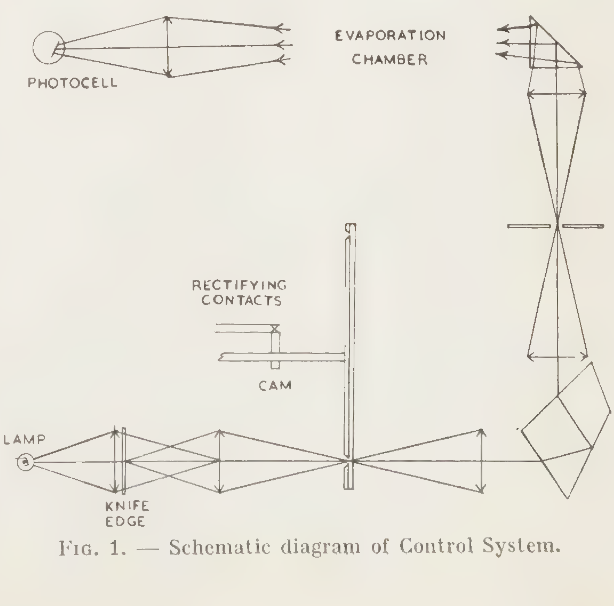

FIG. 1. — Schematic diagram of Control System.

figure 1. The evaporation chamber is a large steel tube with glass ends, and a beam of light from a monochromator passes through the filter under construction and on to a photomultiplier cell. The entrance slit of the monochromator is a portion of a clear narrow annulus defined by a duralumin ring and disc clamped eccentrically on a perspex disc. The slit is illuminated by a wide patch of light from a tungsten lamp, whose output is stabilised by accumulators in parallel. When the perspex disc rotates at about 20 c./sec., the effective entrance slit oscillates at this frequency in the direction of the prism dispersion. The slit is such that when the monochromator is set for λ 6 000 the spectral width and amplitude of wavelength excursion of the beam emerging from the fixed exit slit are both about 100 Å.

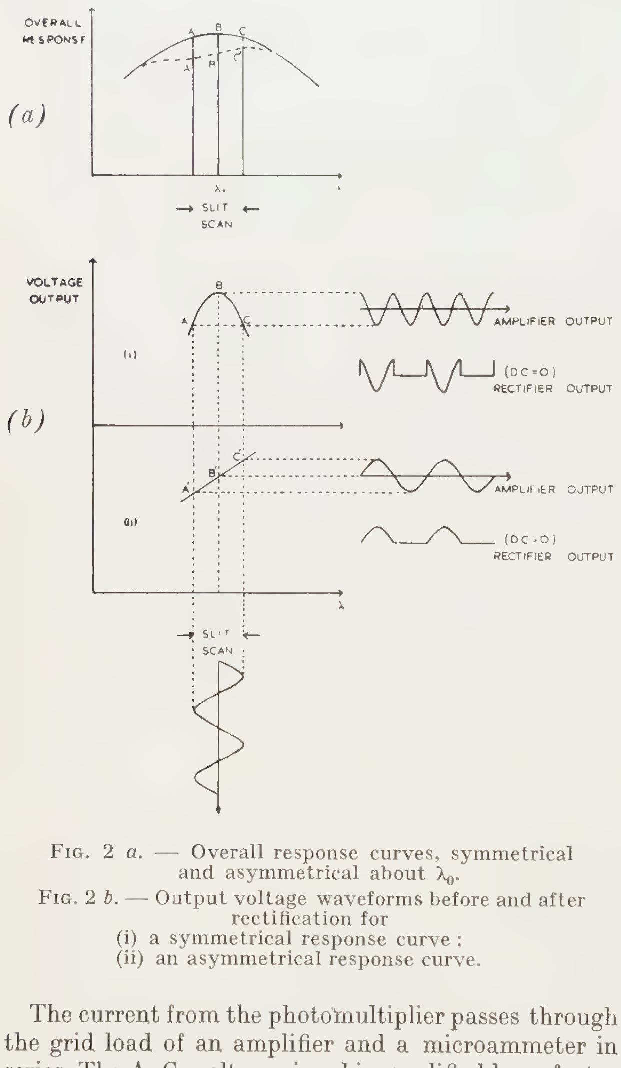

Fig. 2 *a*. — Overall response curves, symmetrical and asymmetrical about λ_0.
Fig. 2 *b*. — Output voltage waveforms before and after rectification for
(i) a symmetrical response curve ;
(ii) an asymmetrical response curve.

The current from the photomultiplier passes through the grid load of an amplifier and a microammeter in series. The A. C. voltage signal is amplified by a factor of about 1 000 and synchronously rectified by contacts on the same shaft as the perspex disc. Any voltage component at the frequency of rotation of the disc gives, after rectification, a D. C. signal which is applied to an *Esterline-Angus* recorder through a cathode follower. Because of the spectral variation of the lamp emission, of the monochromator dispersion and of the photo-cathode sensitivity, the overall wavelength response ([2]) of the system has a maximum at about λ 5 200, falling to 10 % of this value at either end of the visible spectrum. An adjustable inclined knife-edge is placed in the optical system and is projected on to the oscillating slit by a lens. The patch of light illuminating the slit is then wedge-shaped, and as the slit moves across the wedge, the illuminated length varies linearly. This variation may be made to compensate the change in spectral response in any required region by adjusting the inclination of the knife-edge. If the overall response curve has a stationary point at the mean wavelength, λ_0, of the slit scan, the amplifier voltage output contains no component at the scanning frequency [fig. 2b (i)], and after synchronous rectification there is zero D. C. output. The knife-edge is used to establish this condition for the required wavelength before any films are deposited.

As the evaporation of the first (high refractive index) dielectric layer proceeds, it causes a dip to appear in the spectral distribution of light falling on the photomultiplier and to travel in the direction of increasing wavelength. Figure 2a shows how the symmetry of the response curve about the control wavelength, λ_0, is disturbed during the deposition. This asymmetry results in a component of the output voltage at the scanning frequency [fig. 2b (ii)] and a consequent D. C. signal at the recorder which increases and then decreases to zero as the dip becomes symmetrical about λ_0, i. e. when the film has optical thickness $\lambda_0/4$. The process is repeated for subsequent layers.

Figures 2b illustrate the output voltage waveforms obtained with simplified response curves ; in general other harmonics of the scanning frequency are also present, but these are all even if the response curve is symmetrical about λ_0, and give no D. C. component after synchronous rectification.

The amplitude of the D. C. signal is greater during the deposition of a layer of high refractive index than during that of a low index layer, and the two signals differ in sign. Figure 3 shows the recorder trace obtained in the preparation of a filter consisting of two 6-layer films and a second-order spacer of ZnS. The signal is seen to be small for the first two layers, and the microammeter reading which shows the maximum or minimum of transmission is normally used for the control of these layers. For subsequent layers the recorder signal allows of much greater accuracy of control. The largest signals are seen to be obtained for the spacer and the layers subsequent to it. Thus, monitoring a single surface on which the layers are successively deposited gives an accuracy of thickness con-

(2) With the described optical system, the « overall wavelength response » refers to the variation of photomultiplier current as a function of the mean wavelength given by the monochromator.

trol, for the most important layers, several times greater than that obtained by the use of a separate control surface for each layer, in addition to being experimentally much more convenient. At this stage, direct transmission measurements on the filter are misleading unless an exploring beam is used whose spectral width is small compared with the transmission band of the filter. Control plates separate from the filter have been used by most other workers to follow the deposition of these later layers.

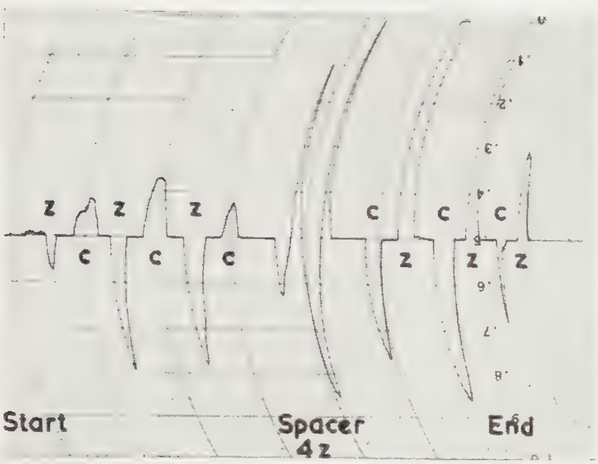

FIG. 3. — Recorder trace.

One unsatisfactory feature of this system of control is that throughout the filter construction the amplifier waveform contains a large component at twice the oscillation frequency. It is difficult to design a high gain amplifier which will deal with this signal without distortion of the wanted signal. The difficulty may be eliminated completely by a different design of wavelength modulation.

If the control beam is switched abruptly between two slightly separated wavelengths, symmetrically placed about λ_0, the output from the photocell is a square wave, whose amplitude is determined by the difference in transmitted intensity at these wavelengths; this is a measure of the asymmetry of the wavelength response curve. This form of modulation will shortly be tried in our system, although the present arrangement is adequate for most requirements.

Results. — About sixty all-dielectric filters have been prepared in this way, of construction ranging from two sets of three layers about a first-order cryolite spacer, to sets of ten layers about a second-order zinc sulphide spacer. The accuracy of control was measured by preparing sets of five or more filters at the same control setting. The position of the pass-band of each was measured and the spread of these values indicates the accuracy of thickness control. The probable error of construction of a single filter, obtained from these results, is ± 10 Å for second-order and ± 15 Å for the first-order filters (at λ 6 500). This compares favourably with the error of ± 260 Å to be expected from D. C. transmission measurements using JARRETT's results, or 130 Å error from A. C. transmission measurements using results obtained by our earlier technique. When the control monochromator was set on a particular spectral line (λ 6 562) the mean wavelength of 21 filters produced was 6 580 Å, showing that no appreciable systematic error was introduced in construction. All the filters were made with a pass-band near λ 6 500 where the control monochromator has low dispersion. For pass-bands in the blue the method may show some improvement in accuracy since the spectral range scanned will be decreased.

TABLE 1. — Half-widths and peak transmissions for various types of filters.

Type of Filter	Wavelength of Pass-Band	Half-width (Å)	Peak Transmission (%)
3-C-3	5 185	380	90
5-C-5	4 750	110	85
6-Z-6	6 565	65	90
7-C-7	5 200	40	75
8-Z-8	6 500	35	80
10-Z-10	6 600	20	50

Half-widths and peak transmissions of representative filters of different types. In describing the type of filter the figures denote the number of quarter-wave layers in each reflecting film and the letter indicates the substance used for the spacing layer (C = Cryolite, Z = Zinc Sulphide).

The half-widths and peak transmissions of these filters, some of which are listed in Table 1, were not markedly better than those of filters prepared by our former, less accurate, technique. This confirms our opinion that the reflectivity of multi-layers is not sensitive to errors of a few per cent. in the thickness of individual layers. However, our present technique is still advantageous for making reflecting films as distinct from filters, since it obviates the use of control plates and reduces the skill needed in locating maxima and minima.

The advantages of this control system are thus seen to be ease of construction of multilayer reflecting films and interference filters without the use of control plates, and precision location of filter pass-bands to such an accuracy that practically every filter made is useful at its predesigned wavelength.

Surface Uniformity. — In most applications the useful area of a filter is limited to that part over which the position of the pass-band varies by less than its half-width. Thus, if full advantage is to be taken of the extremely narrow pass-bands which can be achieved by the use all-dielectric films [6], attention must be paid to the uniformity of the layers which constitute the filter, to that of the spacer in particular.

Our measurements on filters produced by the con-

ventional method of vacuum evaporation from a source placed below a stationary glass plate show that the thickness of the layers varies approximately as $(\mathrm{Cos}\ \theta)^{2.6}$, where θ ($< 7^{\circ}$) is the angle between the normal to the plate which passes through the source and the line joining the latter to the point on the filter at which the thickness is measured. The second column in Table 2, based on this distribution and an evaporation distance of 18.3 cm, illustrates how the useful area of a filter then varies with its half-width.

Table 2. — Variation with half-width of the useful area of a filter.

Half Width (Å) at λ 5000	Radius of Useful Area (cm)	
	Stationary plate Evap. Dist. = 18.3 cm	Rotating plate H = 18.3 cm R = 11.1 cm
20	0.9	3.5
30	1.1	7.2
50	1.5	7.7
70	1.9	8.1
90	2.2	8.4

To improve the surface uniformity we have used a system similar to that described by Dufour [7], whereby the plate is rotated at a height H above a source fixed at a distance R from the axis of rotation.

Figure 4 shows the thickness distribution obtained empirically for various source-plate distances (H), and although these follow the same pattern as the curves calculated for a source obeying Knudsen's law [8], the value of H obtained for the optimum distribution is not in accordance with this theory.

Nevertheless, from a comparison of the two sets of figures in Table 2 it is clear that the useful area of filters has been considerably increased by using this method. The ultimate performance is at present limited by the dimensions of our chamber and minor inconsistencies in the distribution from a given source which make it unprofitable to use a fixed diaphragm to reduce the residual non-uniformity.

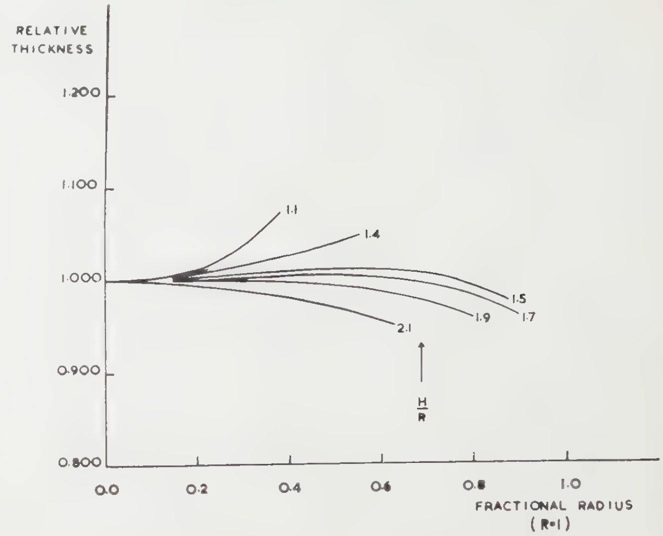

Fig. 4. — Experimental thickness distribution on the rotating plate.

Acknowledgments. — We would like to acknowledge the constant advice of Dr W. L. Wilcock and of Dr H. J. J. Braddick who suggested the form of scanning adopted in our control system. We are also grateful to Dr G. F. J. Garlick of the University of Birmingham who has very kindly supplied us with zinc sulphide specially treated for vacuum evaporation. One of us (P. H. L.) is indebted to the D. S. I. R. for a maintenance grant.

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Lettres à l'Éditeur

Influence d'un mydriatique sur l'effet Stiles-Crawford *

Mlle Lucia Ronchi

En 1933 [1] Stiles et Crawford ont découvert que la luminance d'un objet n'est pas la même si les rayons lumineux passent par le centre de la pupille de l'œil ou par le bord. Ils en ont déduit que tous les rayons arrivant sur la pupille de l'œil, n'avaient pas la même efficacité visuelle [2, 3].

On considère un œil qui observe 2 plages lumineuses juxtaposées, provenant d'un objectif qui fait converger sur la pupille 2 faisceaux issus d'une même lampe ; l'une des images est fixe, au centre de la pupille, tandis que l'autre peut se déplacer d'un bord à l'autre. L'observateur fait alors des égalisations de luminance entre les 2 plages pour différentes positions de l'image mobile. Les 2 images sont d'abord réunies au centre de la pupille où l'observateur fait une mesure de la luminance B_0, puis l'image mobile est amenée à la distance r du centre, soit B_r la luminance mesurée en ce point, on appellera « facteur d'efficacité pupillaire » le rapport

$$\eta = \frac{B_r}{B_0}\,.$$

La courbe donnant le logarithme de η en fonction de la distance au centre de la pupille est donnée approximativement par la formule

$$\log \eta = -\alpha\, r^2$$

où l'on peut regarder α comme la mesure de l'effet étudié ; α dépend du niveau de luminance, de l'observateur et, s'il s'agit de lumière monochromatique, de la longueur d'onde.

D'après Stiles, Crawford et Flamant [4] et d'autres observateurs, α ne doit pas être supérieur à 0,10 mm^{-2}, toutefois les mesures de l'effet intégral faites par d'autres chercheurs donnent des résultats très différents. Du fait de ce désaccord, Toraldo et Sbrolli [5] furent amenés à supposer que les stimuli cohérents venant de points différents de la pupille et se superposant sur la même aire rétinienne, ne doivent pas être additifs.

(*) Cette recherche a été possible grâce au contrat N. AF 61 (514)-634 C passé avec l'*European Office of the Air Research and Development Command*, Bruxelles et l'*Istituto Nazionale di Ottica*, Arcetri-Firenze.

L'auteur et d'autres chercheurs de l'*Istituto Nazionale di Ottica* ont entrepris l'étude de ce phénomène pour expliquer les désaccords concernant la valeur numérique de α.

Le schéma de l'appareil utilisé est donné par la figure 1 : il s'agit d'un sensitomètre pupillaire dont la description sera donnée ailleurs.

Le champ visuel a un diamètre angulaire de 30′ seulement, ce qui donne la certitude de travailler en vision fovéale. La source de lumière est une lampe à sodium et la luminance est de 10^{-4} stilb environ.

Le niveau étant faible, les expériences sont faites sans dilatation artificielle de la pupille ; le diamètre de celle-ci est sans doute supérieur à 5 mm.

Chacun des 2 points images qui se forment sur la pupille de l'œil a un diamètre inférieur à 0,1 mm.

L'œil étant adapté à l'obscurité, l'effet Stiles-Crawford a été mesuré le long d'un méridien horizontal. La valeur de α obtenue est environ 0,11 mm^{-2}.

Quand nous avons répété les mêmes observations, l'observateur ayant la pupille dilatée avec un mydriatique, les résultats ont été tout à fait bouleversants. Le *Simpamine collirium*, solution stérile isotonique à 2 % de sulfate de béta-phényl-isopropylamine en liquide lacrymal a été employé.

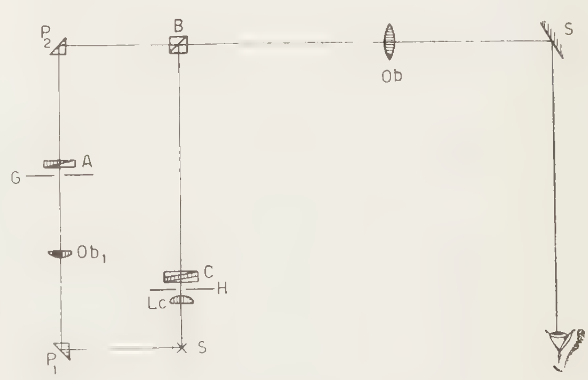

Fig. 1.

Dans la figure 2 le logarithme du facteur d'efficacité est représenté en fonction de la distance au centre de la pupille. Les cercles se rapportent aux mesures obtenues entre une 1/2 heure et 2 heures après l'instillation. Les croix et les triangles se rapportent aux mesures obtenues 5 heures et 6 heures 1/2 respectivement après l'instillation. Les points avaient été obtenus le jour précédent.

On remarque la décroissance rapide de l'effet Stiles-Crawford obtenu sur notre observateur quand le mydriatique est employé. La valeur numérique de α décroît jusqu'à 0,05 mm^{-2}.

Nous avons obtenu les mêmes résultats lorsque nous avons mesuré l'effet Stiles-Crawford avec un diaphanomètre [6]. Dans ce cas nous avons employé de la lumière blanche et différents tests objets.

On atteint la valeur minimum de α quelquefois après 1 heure et d'autres fois une demi-heure après l'instillation. Il est évident que des facteurs inhérents aux conditions nerveuses générales de notre observateur doivent jouer un certain rôle dans ce phénomène.

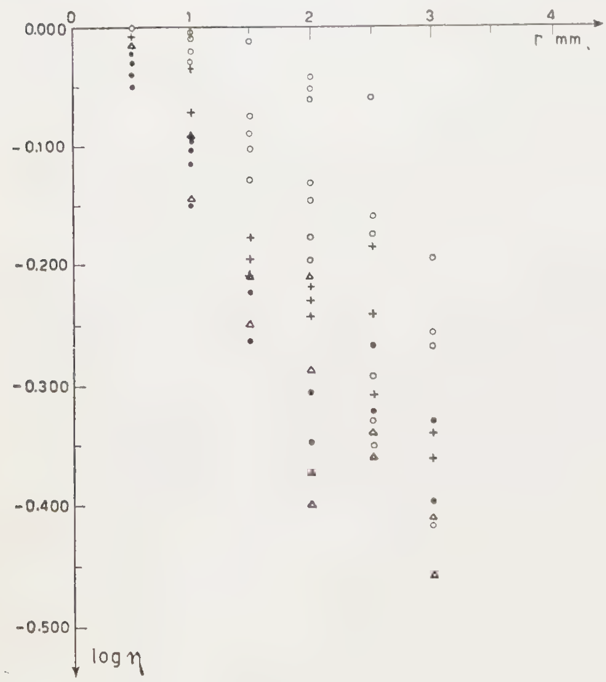

Fig. 2.

Considérons maintenant la figure 3 où log η est porté en fonction de la distance r au centre de la pupille. On a obtenu la courbe (a), sans faire usage du mydriatique ; la courbe (b) une demi-heure après l'instillation ; les courbes (c), (d), (e) respectivement 50 minutes, 1/2 heure et 2 heures après l'instillation.

Dans ce cas on atteint l'effet maximum une demi-heure après l'instillation, ces mesures se référant à la méridienne verticale (en haut).

Dans la figure 4 on porte en abscisses le temps t après l'instillation (en heures) ; en ordonnées le logarithme η.

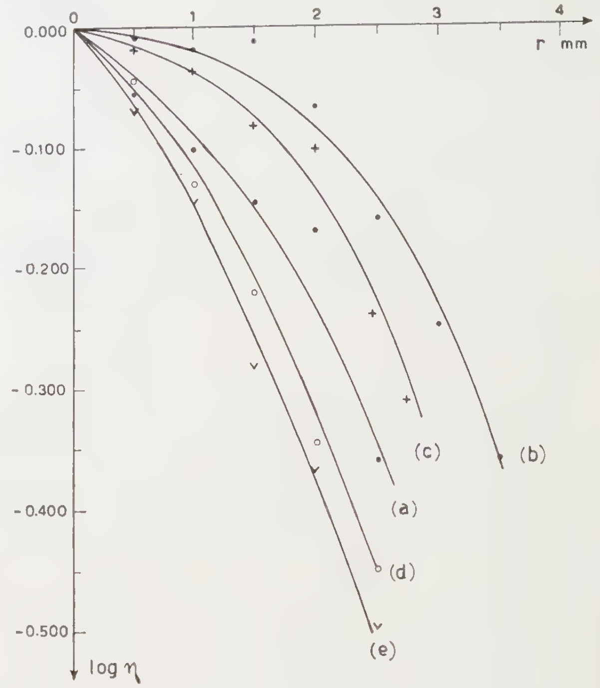

Fig. 3.

L'observateur fait des égalisations de luminance entre les 2 plages produites par les 2 faisceaux, l'un pénétrant au centre de la pupille tandis que l'autre est déplacé de 2 mm.

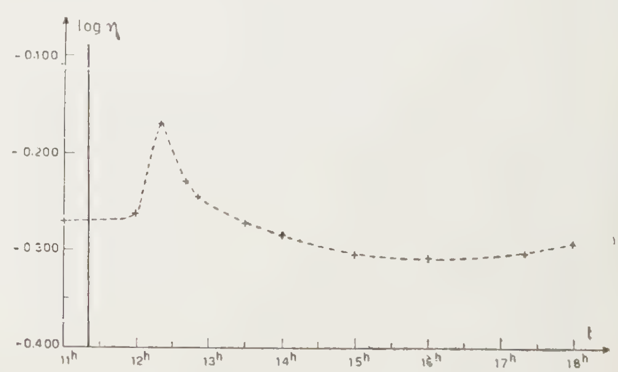

Fig. 4.

L'effet maximum est maintenant atteint 50 minutes environ après l'instillation.

Ces résultats demanderaient à être complétés par de nouvelles expériences, répétées sur un grand nombre d'observateurs ; nous ne pouvons pas encore tirer de conclusions, ni même avancer une explication théorique.

Nous n'avons pas jusqu'ici de référence indiquant que d'autres chercheurs aient déjà observé cette dimi-

nution de l'effet STILES-CRAWFORD. Peut-être cette diminution pourrait-elle être utile pour expliquer les désaccords concernant l'additivité mentionnés ci-dessus.

Note : STILES et CRAWFORD, en 1933, n'ont pas mentionné ce fait. Toutefois il faut remarquer que le mydriatique (euphthalmine hydrochloride) employé par ces auteurs était tout à fait différent du nôtre.

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Two-dimensional coding of optical images

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1. **Introduction**. — Transfer of optical images without the use of lenses or mirrors has been described by the author [1, 2] and by H. H. HOPKINS and N. S. KAPANY [3]. The method given in these publications makes use of bundles of transparent fibres, each preserving the light by internal total reflection. On the entrance surface of such a bundle an image is formed by an auxiliary lens system. The image is divided into small elements by the small entrance surfaces of the individual fibres. When these are sufficiently optically isolated from one another, the end surface of the bundle emits the image, with a grain corresponding to the diameter of the fibres. Possible

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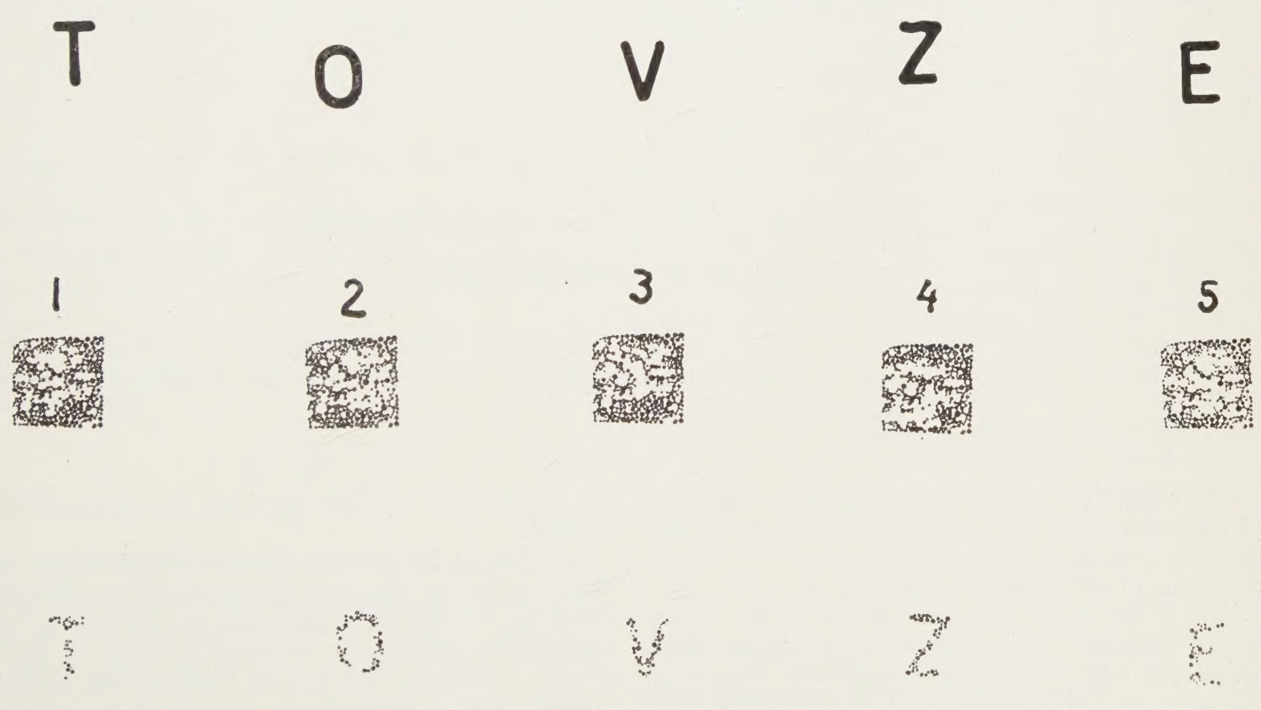

FIG. 1. — Coding and decoding of five images with thesame apparatus.

applications lie in the field of cystoscopes, gastroscopes, etc.

Technical difficulties as yet have held up the manufacture of a practical instrument. An other way of using the same fundamental idea may be described here.

2. **Two dimensional image coding**. — When the fibres in a bundle are not laid in a regular manner, the result at the exit surface is not recognizable. In fig. 1 we show the exit « images » (second row), when the letters given in the first row are presented at the entrance surface.

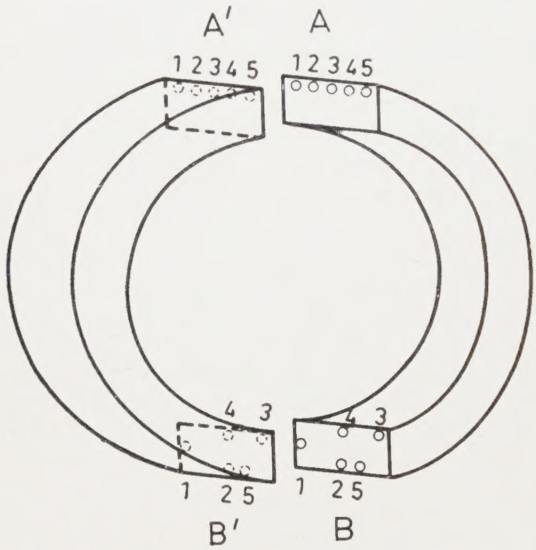

Fig. 2. — The manufacture of two bundles of fibres, giving the same coding.

By making a print of the results and applying the prints one after another at the end surface, taking care to put them in exactly the right position, the images of the third row are obtained, when the direction of illumination is reversed.

Even with a comparative small number of elements (about 400 glass fibres, thickness 0.2 to 0.4 mm, length 25 cm) a reasonable coding is obtained.

If it is necessary to make two corresponding bundles as is often the case, the following procedure may be used. A ring is formed from the fibres by winding them at one point of the circumference (A in fig. 2) in a regular way, making the loops of different lengths (in an irregular succession of loop lengths) and cutting the ring in two places, one near A, the other far from A.

The section of the fibre indicated by 1 returns after one loop at A′ with a translation of one thickness and continues as nr. 2 in A. The fibre follows a screw line (for each layer). Thus, taking into account that the sections at B and B′ are identical and that the section of the bundle at A′ is identical with that at A, but is shifted by one fibre thickness, the two halves can be used for the same coding and decoding.

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Comparaison des différents montages à miroirs utilisés en spectrophotométrie.

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En raison du développement actuel des spectrophotomètres à miroirs, il est intéressant de définir quel sera le meilleur schéma optique à concevoir dans chaque cas. Cette étude uniquement basée sur les propriétés géométriques des spectrophotomètres, a pour but de comparer des instruments dont les caractéristiques photométriques seront supposées identiques. Par suite, seules les aberrations dues à l'optique à miroirs seront envisagées, étant donné que ce sont elles qui diminuent la pureté spectrale du rayonnement transmis.

En utilisant la méthode de calcul décrite dans un article précédent [1], nous pouvons chiffrer l'importance des aberrations du 3e ordre qui seules nous intéressent, le champ d'un spectrophotomètre et généralement son ouverture restant toujours très faibles.

Nous allons d'abord étudier le cas où les miroirs utilisés sont de forme sphérique.

Étude des monochromateurs simples. — Un monochromateur simple se compose d'un système dispersif situé entre deux miroirs sphériques, l'un servant d'objectif collimateur et l'autre d'objectif de chambre, ces deux miroirs pouvant former un ensemble placé suivant un montage en Z ou un montage croisé.

C'est la valeur géométrique de chacun de ces deux montages que nous allons comparer.

Afin de simplifier les résultats, admettons que les spectrophotomètres aient un grandissement aux fentes égal à — 1 et que leur pupille de sortie de forme circulaire soit située à l'infini.

Le calcul nous montre alors que le montage en Z des miroirs sphériques supprime la coma et que si l'ouverture géométrique α n'est pas trop grande, nous pouvons négliger l'aberration sphérique devant un défaut plus important dû à l'astigmatisme. En effet, la focale tangentielle subit une rotation proportionnelle à la hauteur h de la fente utilisée provoquant un élargissement de l'image finale ayant pour valeur

$$e_1 = \frac{3}{2} h \, i \, \alpha$$

(i étant l'angle d'incidence sur les miroirs du rayon moyen considéré).

Par contre, dans le cas du montage croisé le calcul nous indique que la focale tangentielle ne tourne plus et que nous pouvons négliger l'aberration sphérique devant la coma qui est seule responsable de l'élargissement de l'image de la fente, élargissement qui prend alors la forme

$$e_2 = \frac{3}{8} f \, i \, \alpha^2$$

(f étant la distance focale commune aux deux miroirs sphériques).

Comparons les deux quantités e_1 et e_2 obtenues et cherchons pour quelle valeur de h la diffusion due aux aberrations est la plus faible suivant le montage utilisé.

En égalisant e_1 et e_2, nous observons immédiatement que la diffusion dans chaque cas sera identique pour la valeur

$$h = \frac{D}{4}$$

(D étant le diamètre de la pupille du monochromateur).

Un montage en Z est donc préférable lorsque l'on utilise une hauteur de fente plus petite que le quart du diamètre pupillaire, pour des hauteurs de fente plus grandes un montage croisé doit être utilisé.

Nous savons qu'il est possible dans chaque cas de supprimer ces diffusions, car l'élargissement de l'image dû à un montage en Z peut être annulé en donnant aux fentes une courbure telle qu'en chaque point les focales tangentielles soient tangentes à ces fentes ; la coma d'un montage croisé peut également être annulée en utilisant les montages préconisés par Czerny & Plettig [2], consistant à produire le faisceau parallèle tombant sur le système dispersif par une combinaison de deux miroirs sphériques inclinés l'un sur l'autre [3].

Les corrections que nous venons de signaler ont des inconvénients : dans le montage en Z cette correction n'est pas toujours possible du fait qu'il est très difficile d'éclairer des fentes courbes en utilisant des sources droites ; dans le montage croisé cette correction oblige deux réflexions supplémentaires, diminuant ainsi la qualité photométrique du monochromateur.

Notons au passage que l'on peut considérer le montage de Littrow comme un montage croisé, mais le calcul [1] nous indique que dans ce montage il existe un ensemble d'aberrations composé de coma et de rotation de la focale tangentielle. Ce montage doit donc être évité le plus possible.

Étude des monochromateurs doubles. — a) *A dispersion soustractive.*

Dans le cas des monochromateurs doubles à dispersion soustractive, ce qui vient d'être dit reste valable, mais toutefois une remarque s'impose.

Les deux parties du monochromateur sont symétriques par rapport au plan contenant la fente centrale, donc en utilisant deux montages croisés la coma est annulée à la fente de sortie et l'image finale est bien meilleure que celle donnée par deux montages en Z, la pureté du rayonnement étant toujours celle déterminée par les calculs précédents.

b) *A dispersion additive.*

Pour ces instruments, les deux éléments sont symétriques par rapport à la fente centrale et les résultats concernant les monochromateurs simples restent valables dans ce cas précis.

Cette étude comparative des spectrophotomètres permet donc de rendre compte du montage convenant le mieux à l'usage envisagé. La règle établie n'est pas impérative, mais il est recommandé de déterminer la qualité de l'appareil soit par le calcul, soit par l'expérience à l'aide d'un montage préalable, lorsque les fentes utilisées ont des hauteurs plus grandes que le quart du diamètre pupillaire.

Remarquons enfin que le fait d'avoir négligé l'influence de l'aberration sphérique, nous permet d'appliquer ces résultats dans le cas où l'on utilise des miroirs paraboliques.

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Information

L'Institut d'Optique de l'Université de Rochester organise un Symposium sur les *Problèmes posés par la formation et l'évaluation de la qualité des images optiques*, du 15 au 18 juin 1955.

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